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Unpacking the Connections Between Fractions and Algebra: The Importance of Fraction Schemes and Units Coordination

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ABSTRACT
Researchers from multiple disciplines have found that fractions and algebra knowledge are linked. One major strand of research has identified children's *units coordination* structures as crucial for success with fractions and algebra via multiplicative reasoning, whereas a second strand of research points to *magnitude knowledge* as a central conceptual structure supporting fractions and algebra knowledge. In an online study with 59 eighth graders, we compared the associations of these two foundational factors with fractions and algebra knowledge by assessing students' units coordination skills, magnitude knowledge, and several general cognitive skills alongside multiple aspects of fractions and algebra knowledge. In regression models controlling for math anxiety and general cognitive skills, units coordination and fraction schemes were better predictors of students' algebra knowledge than were fraction magnitude or fraction arithmetic skills. We replicated findings from previous studies that showed units coordination and fraction schemes were closely related and showed for the first time that units coordination was significantly associated with fraction number line estimation and fraction arithmetic scores, even after accounting for general cognitive skills. We emphasize the need for interdisciplinary teams to continue investigating relations among these constructs to further our understanding and ultimately support student learning of fractions and algebra.

KEYWORDS
Algebra; fraction schemes; magnitude; middle school; units coordination

Relations Between Units Coordination, Fraction Understanding, and Algebra Knowledge

Understanding fractions is essential for many everyday tasks and foundational to many areas of advanced science and mathematics (Chen, 2013; Torbeyns et al., 2015). Research suggests that fractions knowledge is particularly important for success in algebra, which is a "gatekeeper" in U.S. school mathematics, with important consequences for social equity (Gamoran & Hannigan, 2000; Matthews & Fuchs, 2020; Moses & Cobb, 2001). Multiple quantitative studies have shown that children's fractions knowledge is a robust predictor of their algebra knowledge (Barbieri et al., 2021; Booth & Newton, 2012; Booth et al., 2014; DeWolf et al., 2015; Hurst & Cordes, 2018a; Liang et al., 2018; Siegler et al., 2012). A largely separate body of work using qualitative methods has also shown that children's reasoning about fractions relates to their algebraic reasoning (e.g., Empson et al., 2011; Eriksson & Sumpter, 2021; Hackenberg, 2013; Tunç-Pekkan, 2008). The current study was motivated in part to explore the extent to which these aspects of fractions knowledge and algebraic outcomes are associated with units coordination.

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This study's design and hypotheses were preregistered on OSF. All deidentified data, analysis code, and nonproprietary research materials are available by emailing the corresponding author and will be made available on OSF upon acceptance.

Although converging evidence for a fractions-algebra connection has emerged from multiple strands of research, the mechanisms that link fractions knowledge and algebra knowledge remain unclear (Viegut, 2020). One plausible explanation is that fractions and algebra knowledge are supported by the same underlying precursor skills. Quantitative studies have shown that the fractions-algebra association remains even after statistically controlling for related abilities such as whole number arithmetic, working memory, vocabulary, and reading fluency (Siegler et al., 2012). However, other important precursors that support both fractions and algebra understanding have not been included in these studies. Specifically, one major strand of research has identified children's *units coordination* structures as crucial for fractions and algebra (e.g., Hackenberg, 2007). A second strand of research points to *magnitude knowledge* as a central conceptual structure supporting fractions knowledge (Moss & Case, 1999; Siegler, 2016) and algebra knowledge (Booth & Newton, 2012; Booth et al., 2014). In the current study, we aimed to directly compare the influences of these two foundational factors on different aspects of U.S. eighth graders' fractions and algebra knowledge.

Multiple Aspects of Fractions Knowledge Support Algebra Knowledge

Understanding fractions involves understanding multiple overlapping subconstructs (e.g., Behr et al., 1983; Kieren, 1980). These include thinking about a fraction as *part-whole* comparison (e.g., ¾ represents 3 slices out of a whole pizza with 4 slices), as *quotient* (e.g., ¾ represents 3 divided by 4), as *ratio* (e.g., ¾ represents 3 apples : 4 oranges), as *operator* (e.g., ¾ represents a scaling factor that one can multiply by 4 to get 3), or as *measure* on a number line (e.g., ¾ represents the distance that is ¾ of the way from 0 to 1). Unfortunately, prior studies documenting a connection between early fractions knowledge and later algebra knowledge have rarely parsed fractions into specific subdomains (e.g., Liang et al., 2018; Siegler et al., 2012). Thus, they do not speak to which aspects of fractions knowledge are most closely related to algebra knowledge, and claims researchers can make about the mechanisms driving fractions-algebra relations remain limited.

Evidence That Fraction Magnitude Knowledge is Related to Algebra Knowledge

Over the past decade or so, many psychologists examining fractions-algebra relations have privileged fraction magnitude, which involves assigning a single holistic value to the bipartite *a/b* symbol (e.g., Barbieri et al., 2021; Booth et al., 2014; DeWolf et al., 2015; Hurst & Cordes, 2018a). Fraction as magnitude roughly corresponds to Kieren's (1980) "measurement" interpretation of fractions and is typically assessed by psychologists in one of two ways: (a) number line estimation (NLE), in which students estimate the position of fractions on unmarked number lines (most typically from 0–1), or (b) comparison tasks, in which students decide which of a pair of fractions is larger. Even after accounting for other related mathematical and cognitive skills, fraction magnitude knowledge is associated with students' fraction arithmetic skill (Bailey et al., 2017; Siegler et al., 2011) and overall math achievement (Torbeyns et al., 2015).

This research has repeatedly shown fraction magnitude knowledge to be closely related to algebra knowledge. Performance on 0–1 fraction NLE uniquely predicted concurrent or one-year-later algebra equation solving accuracy and knowledge of equation features (e.g., negative sign, equals sign, like terms) in four separate studies of middle schoolers (Booth & Newton, 2012; Booth et al., 2014; DeWolf et al., 2015; Mou et al., 2016). Children's ability to compare fractions to other types of numbers (i.e., whole numbers and decimals) also significantly predicted pre-algebra knowledge among 4th–7th grade children (Hurst & Cordes, 2018a).

Evidence That Fraction Arithmetic Knowledge is Related to Algebra Knowledge

Other studies suggest that *fraction arithmetic* skills are related to students' algebra knowledge, especially algebraic equation solving. For example, M. Hurst and Cordes (2018b) found that only fraction arithmetic – *not* magnitude knowledge – significantly predicted adults' algebra performance. However, evidence about the strength of associations between fraction arithmetic and algebra is mixed. Fraction arithmetic was a stronger predictor of year-long algebra learning than fraction magnitude knowledge for Algebra I students in one study (Barbieri et al., 2021), but in a separate study was not a significant predictor of algebra knowledge for 7th-grade students when controlling for magnitude knowledge (DeWolf et al., 2015).

Evidence That Fraction Schemes are Related to Algebra Knowledge

Interview and case-based studies have shown that students' *fraction schemes* are also closely related to their algebraic reasoning (Hackenberg, 2013; Hackenberg & Lee, 2015, 2016). Fraction schemes refer to students' cognitive frameworks for experiencing and interacting with fractions (e.g., Hackenberg, 2007; Steffe & Olive, 2010). For example, mature fraction schemes help students understand the multiplicative relations between unit fractional parts (e.g., 1/9), the whole (e.g., one whole is nine one-ninths), and non-unit fractional parts (e.g., 5/9 is five one-ninths, and a whole is 9/5 of 5/9). Researchers have outlined a learning progression of increasingly advanced fraction schemes (e.g., Hackenberg & Tillema, 2009; Norton & Wilkins, 2012; Steffe & Olive, 2010). Early fraction schemes involve only partitioning, whereas later schemes use the operations (i.e., mental actions) of disembedding, iterating, and splitting (Hackenberg, 2013; Norton & Wilkins, 2010). One can use *partitioning* to divide a unit into equal parts and *iterating* to repeat one part to make a larger amount. *Splitting* is a composite operation involving simultaneous partitioning and iterating. *Disembedding* involves "taking a part out of a whole without mentally destroying the whole" (Hackenberg, 2013, p. 539). For example, one can construct 3/5 of a whole by partitioning the whole into 5 equal parts, disembedding one one-fifth part, and iterating that part 3 times.

In the current study, we focus on the operation of splitting and two of the more advanced fraction schemes (i.e., reversible partitive fraction scheme (RPFS) and iterative fraction scheme (IFS)), because these schemes often develop later in middle school (Norton & Wilkins, 2012) and have been shown to be related to algebraic reasoning (e.g., Hackenberg & Lee, 2015; Hackenberg et al., 2017; Lee & Hackenberg, 2014). Students who have constructed an RPFS can use partitioning to solve iterative tasks, such as producing a whole unit when given a fractional part like ⅖ of the whole (Norton & Wilkins, 2012). Beyond situations like this, students who have constructed an IFS can also extend iterating and partitioning to think beyond one whole, such as producing a whole unit when given 7/5 of the whole (Norton & Wilkins, 2012).

Multiple studies show that children who have not yet developed more advanced fraction schemes are likely to struggle as they begin to learn algebra, because many of the mental actions used in fraction reasoning are needed for algebraic problem solving and equation writing (Hackenberg, 2013; Hackenberg & Lee, 2015, 2016). For example, Hackenberg (2013) showed that students with more advanced fraction schemes tended to more easily and accurately write algebraic equations and expressions. Hackenberg et al. (2017) showed that students who had not constructed specific fraction schemes (especially if they had not yet constructed a partitive fraction scheme or an iterative fraction scheme) struggled to represent whole number and fractional relationships between unknowns. Constructing an iterative fraction scheme seems to be especially important for reasoning about and writing equations for multiplicative relationships between unknowns (Hackenberg & Lee, 2015; Lee & Hackenberg, 2014). Other researchers have also argued for a theoretical connection between students' multiplicative reasoning or fraction schemes and their algebraic reasoning (Kaput & West, 1994; Nabors, 2003; Thompson & Saldanha, 2003).

Aggregating prior findings to judge which aspects of fractions knowledge relate most to algebra knowledge is challenging, because studies have used different combinations of fractions measures, with different populations. To the best of our knowledge, no study has measured knowledge of fraction magnitude, fraction arithmetic, and fraction schemes in the same students. Therefore, it is unclear 1) which of these aspects of fractions knowledge most closely relate to algebra knowledge and 2) how these aspects of fractions knowledge relate to each other.

The Role of Units Coordination in Fraction and Algebra Knowledge

In addition to fractions-specific knowledge, we investigate the impact of a more general sort of knowledge – *units coordination* – on fractions and algebra knowledge. Units coordination involves the mental structures that organize hierarchically nested additive and multiplicative relationships and allow individuals to reason about composite units (e.g., Hackenberg, 2007, 2010; Ulrich, 2015, 2016). The development of student thinking about additive relationships, for example, concerns moving from working with single arithmetic units to coordinating multiple number sequences to engage in activities such as double counting to reasoning flexibly about relations among all the components in an additive situation (Ulrich, 2016).

We focus in this paper on units coordination in relation to multiplicative situations. The development of units coordination in this context is typically described in three progressively sophisticated stages (Table 1). A simple units coordination example involves reasoning about the number 4 as four units of ones or alternatively as one unit of 4. A child who is in the first stage of units coordination (see Table 1) has interiorized this construction of 4 (i.e., can take this unit for granted) and can use it to solve problems like "In a classroom, there are 3 rows with 4 seats in each row. How many seats are in the classroom?". A child in Stage 1 might solve this problem using double counting; for example, saying "four [puts up one finger], four more is 5-6-7-8 [puts up a second finger], and four more is 9-10-11-12 [puts up a third finger]. Twelve seats!" (adapted from Ulrich, 2015). In Stage 2, students can take composite units (i.e., a unit of units) as given, so they would very quickly be able to recognize that the classroom with 3 rows of 4 has 12 seats without needing to rely on counting. By Stage 3, students can solve extended problems such as determining how many seats there would be in a school with eight such classrooms by taking one entire classroom as a given unit and seeing the school as a unit-of-units-of-units structure (i.e., 8 classrooms, each containing 3 rows, each containing 4 seats). Students in Stage 2 may also be able to solve this extended problem but would need to rely on constructing the solution in activity, perhaps by drawing it out or writing 12 eight times. Units coordination supports reasoning about many mathematical topics including whole number arithmetic, fractions, and algebra (Boyce & Norton, 2016; Shin et al., 2020; Steffe, 1992).

Units Coordination Supports Fraction Knowledge

Some researchers suggest that difficulties students experience coordinating units underlie many of the difficulties students have making sense of fractions (e.g., Hackenberg, 2007, 2010; Hackenberg & Lee, 2015; Norton & Wilkins, 2010; Steffe, 2002). Boyce and Norton (2016) found that sixth-grade students operating with more advanced stages of units coordination with whole numbers had more advanced fraction schemes and concluded that units coordination and fraction schemes co-develop. This is consistent with findings from teaching experiments that students' units coordination is related to their fraction schemes (Hackenberg, 2007; Hackenberg & Tillema, 2009). Students who cannot yet take three levels of units as given (i.e., have not yet reached Stage 3 of units coordination) struggle to construct and reason with improper fractions (Hackenberg, 2007) and would have difficulty solving problems such as that shown in Figure 1.

Table 1. Stages of units coordination.

Stage	Students' unit structures
Stage 1	Students can take one level of units as given, and may coordinate two levels of units in activity.
Stage 2	Students can take two levels of units as given, and may coordinate three levels of units in activity.
Stage 3	Students can take three levels of units as given, and can thus flexibly switch between three-level structures.

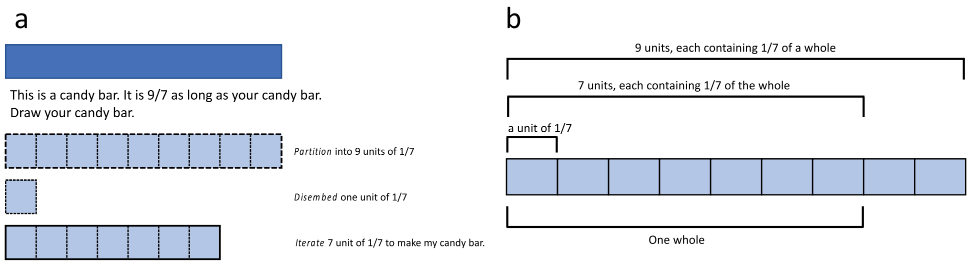

Figure 1. Illustration of units coordination in fraction reasoning. *Note*. (a) A fraction reasoning problem that requires partitioning, disembedding, iterating, and coordinating three levels of units (Hackenberg & Lee, 2015) and a solution strategy. (b) A structure for coordinating units that could be used to solve the problem in A. This bar is structured into 3 levels of units (units of one-seventh, a whole of 7 one-sevenths, and a composite unit of 9 one-sevenths). Adapted from (Stevens et al., 2020).

There is not yet clear evidence from quantitative studies that units coordination is essential for supporting other aspects of fraction knowledge (e.g., magnitude, arithmetic). However, Steffe (2002) and others have found that children must develop complex multiplicative structures with whole numbers and part-whole conceptions of fractions before learning to think about fractions as measures or as operators (e.g., Hackenberg, 2007, 2013; Norton & Hackenberg, 2010; Steffe & Olive, 2010; Wilkins & Norton, 2018).

Units Coordination Supports Algebra Knowledge

Children's units coordination ability has also been shown to be important for their algebraic reasoning (Hackenberg, 2013; Hackenberg et al., 2017; Olive & Caglayan, 2008). Middle school students who could not yet take three-level multiplicative structures as given had difficulty writing algebraic equations to model situations such as "Stephen's iPod cord is some number of feet long. His cord is five times the length of Rebecca's cord" (Hackenberg & Lee, 2015). In contrast, students who could coordinate three levels of units were able to produce an equation (e.g., $S = 5 \times R$) and recognize the reversibility of the multiplicative relation (e.g., $R = S \div 5$). Keeping track of quantitative relations involving unknowns requires students to flexibly coordinate nested multiplicative relations, such as thinking of the length of Stephen's cord as a composite unit of five units, each of which is the length of Rebecca's cord.

Using the lens of the parallel number sequences framework, Zwanch (2022a) found that middle school students' strategies to solve word problems that could be modeled by systems of algebraic equations progressed in sophistication according to their stage in the sequence. Students at lower stages of the number sequences framework tended to use unsystematic guess-and-check strategies. Students at progressively higher stages used more systematic strategies, were better able to keep track of multiple relationships in the problems, and were more apt to make use of algebraic strategies they had learned in school. Middle schoolers' number sequence stage has also been found to relate to their success stating linear generalizations in words or with algebraic symbols (Zwanch, 2022b).

The Role of General Cognitive Skills in the Connections Between Fractions and Algebra

As we investigate connections between different aspects of fractions and algebra knowledge, it is important to also account for the role other cognitive skills may play in both domains. Past research shows that both fractions and algebra knowledge are closely associated with earlier math skills like whole number knowledge (Bailey et al., 2014; Siegler et al., 2012) and general cognitive skills including working memory, spatial skills, and nonverbal reasoning (Jordan et al., 2013; Lee et al., 2009; Möhring et al., 2016; Peng et al., 2016). For example, a greater capacity to hold and manipulate verbal and spatial information in working memory has been shown to be useful for algebra reasoning, particularly for word problems (Lee et al., 2009; Ünal et al., 2022) and for fraction reasoning (e.g., Jordan et al., 2013, Kerrigan, 2022; Norton et al., 2023). In the current study, we measure these skills and statistically account for them, to isolate the relations between specific aspects of fractions and algebra knowledge.

Current Study

Despite evidence from qualitative studies that units coordination is closely related to both fractions and algebra knowledge, no existing quantitative research investigating the fractions-algebra connection has included a measure of students' units coordination. This gap in the literature has at least two important consequences. First, no study has investigated whether the relation between fractions knowledge and algebra knowledge is maintained even after accounting for variability in children's units coordination abilities. Second, it remains largely unknown whether and how individual differences in general cognitive skills like working memory or spatial skills may moderate the links among students' units coordination, fractions knowledge, and algebra knowledge. Fractions and algebra knowledge are closely associated with working memory, executive function, spatial skills, and non-verbal reasoning (Jordan et al., 2013; Lee et al., 2009; Möhring et al., 2016; Peng et al., 2016). Therefore, it is important to account for these relevant skills in order to isolate the relations among units coordination, fractions knowledge, and algebra knowledge.

Using a quantitative approach, we explored relations among middle schoolers' units coordination, algebra knowledge, and different aspects of their fractions knowledge (i.e., schemes, magnitude, and arithmetic) while controlling for general cognitive skills (e.g., working memory, spatial reasoning). The study is the first we are aware of to measure students' fraction schemes alongside their understanding of fraction magnitude and fraction arithmetic and was motivated by a desire to inform specific, actionable models of learning that acknowledge the multiple aspects of fractions knowledge that may relate to algebra knowledge. We asked three research questions: (1) Which aspects of fractions knowledge are associated with units coordination? (2) Is units coordination associated with algebra knowledge when accounting for other related skills? (3) Which aspects of fractions knowledge are most closely related to algebra knowledge when accounting for units coordination and other related math and cognitive skills?

We preregistered this study's design, analysis plan, and four hypotheses on Open Science Framework (https://osf.io/2w3j4/?view_only=35c746ba1c434f8991b869b0a561471c). The hypotheses reported on in this paper are (1) that units coordination and fraction schemes would correlate significantly with each other, even when accounting for covariates and (2) that units coordination and fraction schemes would correlate significantly with fraction magnitude knowledge and fraction arithmetic, because prior work suggests that part-whole fraction understanding is an entry point for other subconstructs of fractions and for fraction arithmetic (e.g., Steffe, 2002). We did not preregister a prediction about which aspect of fractions knowledge would be most strongly related to algebra knowledge, as prior research has alternatively privileged fractions magnitude knowledge (Booth et al., 2014; Siegler et al., 2011) fraction arithmetic skill (Barbieri et al., 2021), or fraction schemes (Hackenberg, 2013) but none has ever accounted for all three factors at once.

Methods

Participants

We recruited eighth-grade U.S. students to participate in this study using social media posts, e-mails to the [blinded for review] community, and online flyers distributed to families through school districts. Participants were told the study would involve three sessions conducted over Zoom from their homes over the course of up to three weeks. Of the 65 students whose parents/guardians consented to the study, three never scheduled any sessions, one could not participate due to internet connectivity problems, one withdrew after completing the first session, and one was excluded due to parent interference. These six students were dropped from all analyses, resulting in a final sample of 59 eighth graders (M_{age} = 14.01 years, SD = 0.34 years; 29 female, 26 male, 2 non-binary, 2 unspecified). One student did not complete the study session focused on algebra due to scheduling problems but was included in all other analyses.

Participants came from 40 schools in Wisconsin, Minnesota, Indiana, and Michigan and identified as White (80%), Asian (3.3%), Hispanic/Latino (3.3%), Black/African American (1.7%), and multi-racial (6.8%). A small number (3.3%) chose not to report their race. We did not collect student-level socioeconomic information.

According to parent/guardian report, 39% of participants were currently enrolled in Algebra I, 22% had already completed Algebra I, and 12% had not yet enrolled in Algebra I. For the remaining 27% of students, the parent/guardian either declined to answer or indicated that they were unsure of their student's Algebra I status.

Measures

Units Coordination

Units coordination skill was measured using an adaptation of Norton et al. (2015)'s assessment. This 7-item assessment, presented to participants one item at a time on a shared screen, asked students to reason about how many times a smaller bar would fit into a larger bar using given information about the relations among three bars of different lengths, as seen in Figure 2. Students used Zoom's "Annotate" feature to draw on the screen and record their answers. Those who initially provided answers with no explanations were asked to draw or use words to explain their reasoning. Students' responses were video recorded, and their drawings were saved as digital images.

Fractions Magnitude

Number Line Estimation. Students were asked to click on a number line on the screen to estimate the magnitude of a given fraction, which appeared above the center of the number line. Students first saw the position of ½ on a 0–1 number line as an example and then completed a practice trial with ½. In test trials, participants judged the positions of 41 fractions, first on 0–1 number lines (18 items), then on 0–2 number lines (13 items), and finally on 0–5 number lines (10 items). Students had up to six seconds to respond to each item.

Fractions for the 0–1 block were 1/360, 1/180, 1/45, 5/118, 1/12, 13/85, 1/5, 3/11, 2/7, 1/3, 83/215, 177/352, 3/5, 5/8, 33/47, 7/9, 5/6, and 146/149 (Booth et al., 2014). Fractions for the 0–2 block were 1/3, 7/4, 12/13, 3/2, 5/6, 5/5, 1/2, 7/6, 3/8, 2/3, 7/9, 1/19, 4/3 (Hansen et al., 2015), and fractions for the 0–5 block were 7/8, 11/7, 13/4, 9/5, 13/6, 7/3, 10/3, 9/2, 19/4, 1/5 (Hansen et al., 2015). Fractions within

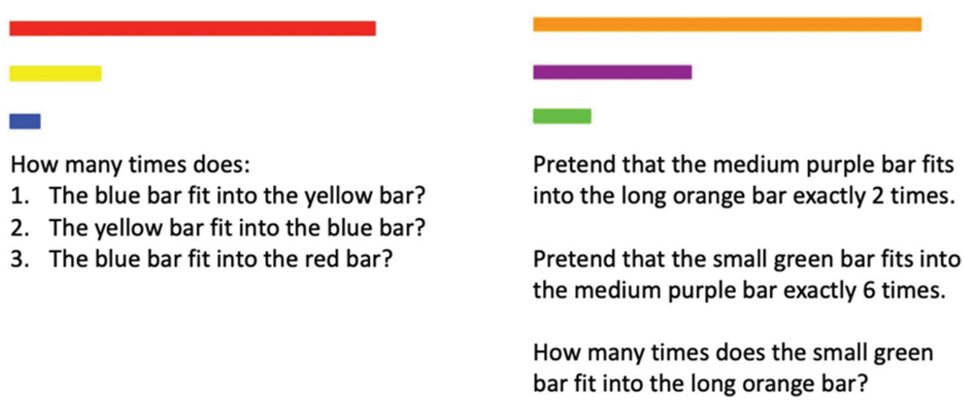

Figure 2. Sample items from the units coordination assessment. *Note*. Task adapted from Norton et al. (2015) paper-and-pencil task.

each block were presented in random order. We scored each estimate by Percent Absolute Error (PAE = (|*estimate* – *actual*|*range* *100) (Siegler & Booth, 2004); and calculated each student's mean PAE. Lower PAE indicates better estimates.

Fraction Comparison. Students completed a 48-item fraction magnitude comparison task indicating via keystroke which of two side-by-side fractions was greater. Participants were instructed to answer as quickly and as accurately as they could. The items included six instances from each of eight different trial types – each designed to elicit specific strategies (Fazio et al., 2016). These trial types included: equal denominator (e.g., 3/7 vs. 2/7), equal numerator (e.g., ¾ vs. 3/5), larger numerator and smaller denominator (e.g., 3/7 vs. 2/9), halves reference (e.g., 2/3 vs. 3/7), multiply for a common denominator (e.g., 2/3 vs. 5/9), multiply for a common numerator (e.g., 4/7 vs. 2/9), large-distance estimation (e.g., 4/9 vs. 1/8), and small-distance estimation (¾ vs. 5/9). Items were presented in random order, and we calculated each student's percent accuracy.

Fraction Arithmetic

Children completed a 19-item test of fraction arithmetic, taken from Kalra et al. (2020). Questions included addition, subtraction, multiplication, and division with fractions and mixed numbers. Students typed their answers but were encouraged to use scratch paper. When students finished or after twelve minutes elapsed, their typed answers were submitted. We calculated each student's percent accuracy.

Fraction Schemes

To assess the sophistication of students' fraction schemes overall, we scored the first 12 items from a 16-item test developed by Norton and Wilkins (2012). The last 4 items were designed to measure units coordination, so these items were excluded from students' fraction schemes scores and instead were included with their units coordination scores. Questions asked students to use partitioning and iterating to draw a whole when given a fractional part and to draw a fractional part when given a whole. As shown in Figure 3, half of the items included bars and half included circular area models. Each question was presented on the screen and read aloud by the experimenter, and students responded by drawing on the screen using Zoom's "Annotate" feature. Students' drawings were recorded in both Zoom video recordings and in screenshots taken by the experimenter after each item.

The fraction schemes test included three groups of four items, with each group designed to measure different operations or schemes: iterative fraction schemes (Figure 3(a)), reversible partitive fraction schemes (Figure 3(b)), and splitting operations (Figure 3(c)). We measured the sophistication and robustness of students' fraction schemes overall by calculating their composite scores on the 12 items across groups.

Algebra

Students completed a 40-item assessment of algebra knowledge (full assessment available on OSF). Students were given 45 minutes to complete the assessment. The algebra measure included 21 items taken from Star and Rittle-Johnson's (2008) assessment of conceptual knowledge (9 items), procedural knowledge (7 items), and flexibility (5 items) in algebra. All but one of these problems were multiple-choice. The final 19 items were open-ended questions that asked students to explain their reasoning, taken from various assessments of algebraic reasoning from math education research (Blanton et al., 2015; Kaput & West, 1994; Rivera & Becker, 2011; Stephens et al., 2021; Swafford & Langrall, 2000). The test included some items involving formal algebraic notation (e.g., Solve the following equation for y: $5(y-2) = -3(y-2) + 4$) and some word problems (e.g., A fourth grade class needs 5 leaves each day to feed its 2 caterpillars. How many leaves would they need each day for 12 caterpillars?). Students were invited to use scratch paper but were told not to use a calculator.

Students completed an average of 35 items (SD = 6.4; range = 17–40) within the time limit, and only 24 of the 59 students (41%) reached the end of the assessment. Therefore, we scored each student's

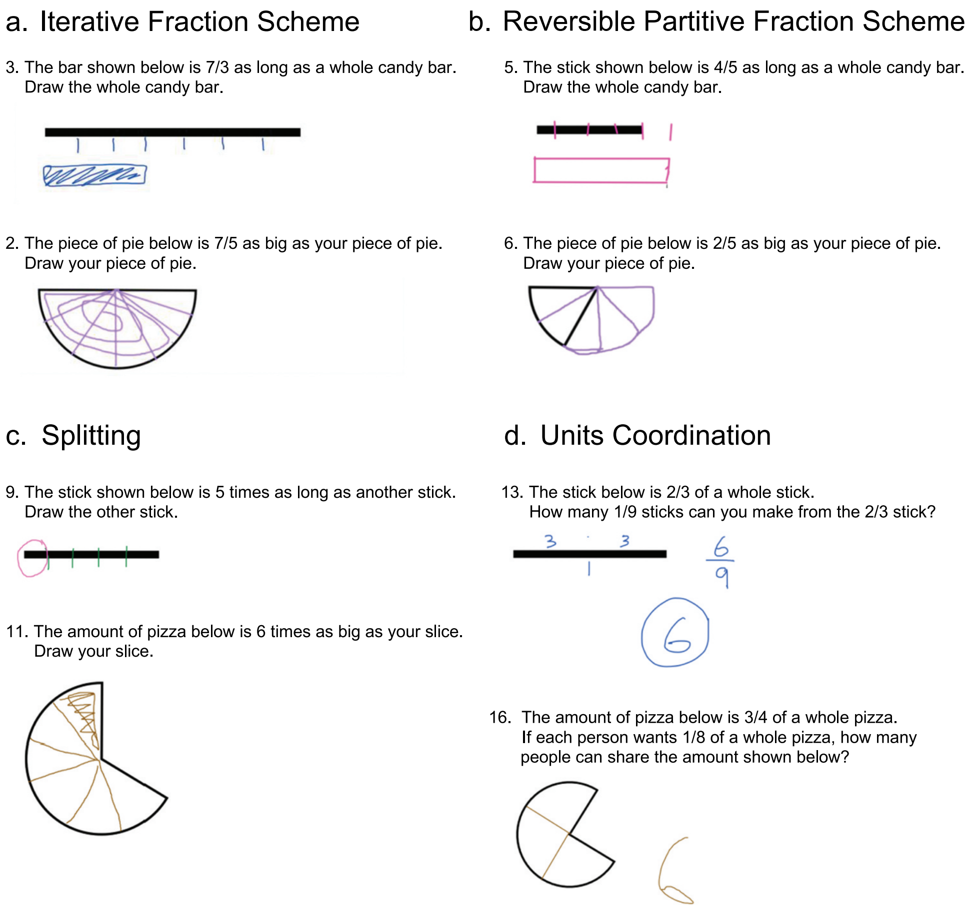

Figure 3. Sample Student responses on the fraction schemes assessment. *Note*. Labels refer to the scheme or operation assessed by each set of items.

performance as the percent of correct answers out of the total number of attempted items. For example, a child who answered 20 questions correctly would receive a score of 50% if they attempted all 40 questions, but a score of 66.7% if they attempted only the first 30 questions within the time limit.

Covariates

Whole Number Estimation. Students estimated the position of 21 whole numbers on unmarked number lines of ranges varying from 0–2 to 0–100. There was no time limit. As we did for fraction number line estimation, we scored each estimate by Percent Absolute Error (PAE = (|estimate – actual|/range *100) (Siegler & Booth, 2004); and calculated each student's mean PAE. Lower PAE indicates better estimates.

Whole Number Arithmetic Fluency. Students completed the Woodcock-Johnson III (WJ-III) Math Fluency subtest, a 3-minute timed test in which they solved up to 160 single-digit addition, subtraction, and multiplication problems. Students saw all problems on one screen and typed their answers. They were instructed to use the "TAB" key to move on to the next problem. Students received one point for each correct answer.

Nonsymbolic Ratio Processing. Students were asked to compare two ratios made from juxtaposed pairs of line segments, following Park et al. (2020). Students were first shown one example problem and then completed four practice trials with feedback about correctness. Next, they compared 40 comparison trials without feedback. Ratios disappeared after four seconds. Stimuli for this task were identical to those used with children by Park et al. (2020).

Nonverbal Abstract Reasoning. Students completed a computerized version of the Ravens Standard Progressive Matrices (Raven, 2003) test to measure their nonverbal abstract reasoning. Students saw incomplete patterns and were asked to select the choice that best completed each pattern. Students were given 20 minutes to complete up to 60 problems, with problems blocked into five progressively more difficult sets of 12 problems each.

Auditory Working Memory. All students completed the Backward Digit Span subtest of the Wechsler Intelligence Scale for Children (WISC-IV). Students heard lists of digits (e.g., 6-9-2-7) and were asked to repeat the list backwards (e.g., 7-2-9-6). After two practice trials, students heard up to 16 lists, increasing in length from two to eight digits. The task stopped when they answered two items of the same length incorrectly. Students received a "Digit Span" score ranging from 2–8, which was defined as the longest list of numbers that they correctly repeated.

Procedure

This study took place over three one-hour sessions on Zoom on separate days. Most families (79%) scheduled all their sessions within a two-week window, and all students completed their sessions within a month. Participating families received electronic gift cards after each session. Each child's parent/guardian filled out the online consent and demographic form before the first session. At session 1, students completed the covariate measures in a fixed order: Backward Digit Span, Raven's Matrices, Whole Number Estimation or Ratio Processing (counterbalanced), Math Fluency, and Math Anxiety. At session 2, students completed all fractions tasks and the units coordination assessment. Tasks were organized into three blocks presented in a counterbalanced order: Arithmetic, Fraction Magnitude (fraction comparison and NLE), and Drawing (fraction schemes and units coordination). Before the drawing tasks, participants practiced using Zoom's drawing tools. If all tasks were not completed within 60 minutes, students were given the option to continue working for up to 15 more minutes. If all tasks were not finished in 75 minutes, the experimenter stopped the session. Finally, at session 3, children completed the 45-minute algebra assessment and completed any make-up tasks (i.e., tasks which had not been started due to time constraints and/or tasks with which they had technical difficulties at sessions 1 and 2).

Approach to Missing Data

Most of the 59 participants completed all tasks. However, 10 students (16.9%) did not complete the full units coordination assessment because they ran out of time or had technical difficulties. Seven students (including five of the aforementioned ten) did not complete the full fraction schemes assessment. For all correlations we used all pairwise complete observations, but for the regression analyses involving units coordination and schemes, we used listwise deletion. Thus, the total sample for analysis was 49 students for all analyses including units coordination and 47 students for all analyses including both units coordination and schemes.

Coding Students' Units Coordination and Fraction Schemes Ability

Two independent coders coded children's written responses to each units coordination item for indicators of Stage 1, Stage 2, or Stage 3 reasoning, following a rubric developed by Norton et al. (2015). After scoring each student's written work, the coders then used the video recording to verify their codes. Each coder then categorized students into one of the three levels. Agreement between coders was 84.7%, and discrepancies were resolved by the first author. Because sorting students only into three levels of units coordination may be collapsing important variance between students, we also scored each item as correct or incorrect, resulting in a continuous score ranging from 0 to 7 for each student. Having a continuous accuracy score for the units coordination assessment allowed us to combine this score with scores from the final four items from the fraction schemes test (see description above), which were designed to measure units coordination with fractions, to make an 11-item units coordination score. We report results using both the categorical stages and the continuous accuracy score.

To score students' fraction schemes, two independent coders assessed students' responses based on their written work in addition to the correctness of their answer, following Norton and Wilkins (2012). Agreement between coders was 85.4%, and discrepancies were resolved by the first author. We scored items as 0 when there was clear contraindication that the student operated in a manner compatible with the scheme/operation being tested (e.g., child drew a bar longer than the original when asked to make a whole candy bar from a 7/5 bar). We scored items as 1 when there was clear indication that the student operated in a manner compatible with the scheme/operation being tested (e.g., child showed the correct number of equal partitions). If we saw neither clear contraindication nor clear indication of the scheme/operation, we scored items as 0.5 (e.g., child estimated a pie slice by scaling the original pie figure, but (possibly because of difficulty using the drawing tools) it was unclear whether their estimate was the proper size). Each student was given a continuous score – the sum of their scores on the 12 problems. We also categorized each student based on the most advanced scheme/operation they had constructed (i.e., the highest level for which the student answered at least half of the items in a manner consistent with that scheme/operation). Therefore, the categories were None, Splitting, Reversible Partitive Fraction Scheme (RPFS), and Iterative Fraction Scheme (IFS).

Analysis Plan

First, we investigated how students' categorical levels of units coordination aligned with students' continuous units coordination scores when considering accuracy alone. We also report the same comparison of continuous versus categorical scoring for fraction schemes. Next, we investigated bivariate correlations between units coordination, aspects of fractions knowledge, algebra knowledge, and covariates. Third, we conducted regression analyses to isolate the relations among units coordination and aspects of fractions knowledge after accounting for other math and cognitive skills. Finally, we compared the contribution of units coordination, different aspects of fractions knowledge, and other covariates to algebra knowledge using regression.

Results

Observed Units Coordination Skills

Among the 49 students who completed the full units coordination assessment, four students (8.2%) showed evidence of operating at Stage 1, 30 students (61.2%) at Stage 2, and 15 students (30.6%) at Stage 3. When we considered accuracy of answers alone as a continuous score, the sample averaged 76.8% correct (SD = 22.3%) on the 11-item assessment. As shown in Figure 4, an Analysis of Variance (ANOVA) showed that accuracy scores significantly differed by stages, $F(2, 46) = 43.0$, $p < 0.001$. Post-hoc pairwise comparisons with Bonferroni's correction showed that students operating at stage 1 (M = 22.7%, SD = 18.9%) had significantly lower accuracy than students operating at stage 2 (M = 76.1%, SD = 14.8%, $p < 0.001$) and students operating at stage 3 (M = 92.7%, SD = 7.8%, $p < 0.001$). There was also a statistically significant difference between the accuracy of students operating at stage 2 and students operating at stage 3 ($p < 0.001$). There was substantial variability in accuracy within each of the three stages (Figure 4), which may suggest that a continuous score can capture differences in

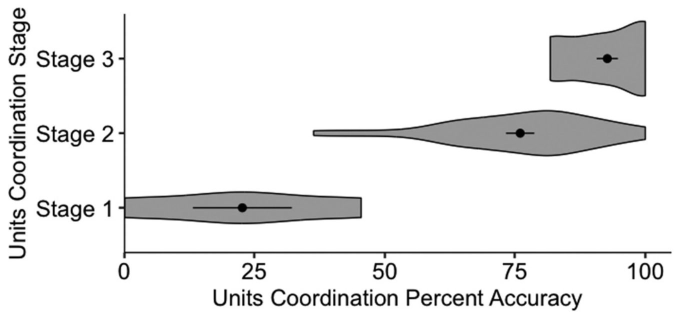

Figure 4. Students' units coordination performance by stage and percent accuracy. *Note*. Four students were coded at Stage 1, 30 students at Stage 2, and 15 students at Stage 3. Stages were associated with accuracy, but there was variability in accuracy within each stage.

students' units coordination skills more precisely than a categorical score. In all following analyses, we run separate models treating units coordination as categorical and continuous, to check the robustness of our findings.

Observed Fraction Schemes Skills

Among the 52 students who completed the full fraction schemes assessment, five students were categorized as "None" because their responses did not show clear evidence of having constructed a splitting operation, reversible partitive fraction scheme (RPFS), or iterative fraction scheme (IFS). Among the remaining students, 26 students showed evidence of having constructed an IFS, 12 showed evidence of having constructed an RPFS (but not an IFS), and 9 showed evidence of having constructed the Splitting operation (but not RPFS or IFS). When we considered performance on the fraction schemes assessment as a continuous score, the sample averaged 70.0% correct ($SD = 21.7\%$) on the 12-item assessment.

As shown in Figure 5, the continuous overall scores for students classified at the 4 stages differed in a pattern that matches the expected developmental progression of fraction schemes. An ANOVA showed that students' stage had a statistically significant effect on their continuous overall score $F(3, 48) = 36.3, p < 0.001$. Post-hoc pairwise Bonferroni comparisons showed that the overall continuous scores of students assigned to each stage were statistically significantly different than the overall scores of

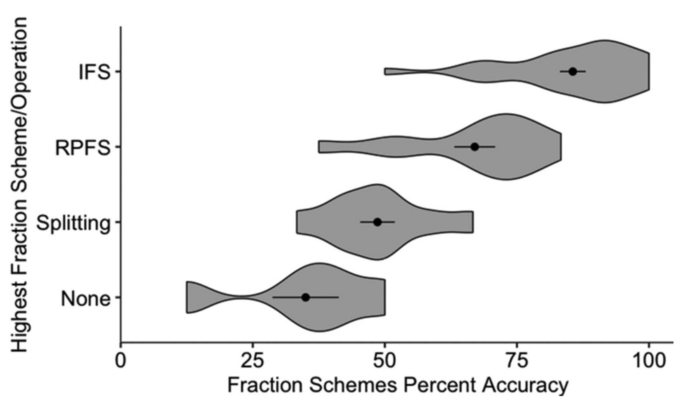

Figure 5. Students' fraction schemes performance by stage and percent Accuracy. *Note*. Five students were coded as "None," nine students as having constructed the "Splitting" operation, 12 students as having constructed the reversible partitive fraction scheme (RPFS), and 26 students as having constructed the iterative fraction scheme (IFS). Stages were associated with accuracy, but there was variability in accuracy within each stage.

students at each of the other stages (all $p < 0.01$), except for those in the "none" and "splitting" stages, which were not statistically significantly different ($p = 0.325$). However, there was also substantial variability within the 4 stages in children's continuous scores (Figure 5), just as we saw for units coordination. Because the continuous score aligns with the categorical levels of fraction schemes and captures additional variability within those levels, we report results using the continuous score for fraction schemes in all following analyses.

Relations Between Units Coordination and Fractions Knowledge

Students' units coordination scores significantly correlated with all measured aspects of their fractions knowledge, including NLE ($r = -.60$, $p < 0.001$), comparison ($r = .45$, $p = 0.001$), arithmetic ($r = .52$, $p < 0.001$), and schemes ($r = .58$, $p < 0.001$). As shown in Table 2, all fraction measures were also significantly correlated with each other, with correlation coefficients ranging from $r = .42$ to $r = -.66$. Units coordination was also correlated with other math and cognitive skills, including Raven's Matrices ($r = .59$, $p < 0.001$), spatial proportional reasoning ($r = .45$, $p < 0.001$), whole number estimation ($r = -.40$, $p = 0.005$), working memory ($r = .34$, $p < 0.001$) and whole number arithmetic fluency ($r = .31$, $p = 0.032$). Units coordination was *not* related to math anxiety ($r = -.09$, $p = 0.53$).

To isolate the specific association between units coordination and different aspects of fractions knowledge, we regressed each fraction test separately on units coordination, controlling for the contributions of math anxiety, whole number estimation, whole number arithmetic fluency, working memory, spatial proportional reasoning, nonverbal abstract reasoning, and age. After accounting for these covariates, units coordination explained significant variance in fraction NLE ($\beta = -.49$, $p = 0.005$), fraction arithmetic ($\beta = .36$, $p = 0.028$), and fraction schemes ($\beta = .37$, $p = 0.023$). We used standardized regressions in all regression analyses, so the beta value can be interpreted as an effect size like Cohen's *d*. For example, our results show that an increase of 1 standard deviation in units coordination was associated with an increase of 0.36 standard deviations in fraction arithmetic. However, units coordination was not significantly associated with fraction comparison after controlling for the covariates ($\beta = .23$, $p = 0.209$). The full regression results for all four aspects of fraction knowledge are reported in Table 3. As shown in Table 3, units coordination had a stronger effect on fraction number line estimation, fraction arithmetic, and fraction schemes than any of the covariate whole number math skills or general cognitive abilities.

We then examined whether children who displayed different units coordination stages performed differently on the fraction assessments using analysis of covariance (ANCOVA), controlling for the same set of covariates as in the regression analyses. Units coordination stages explained significant variance on all fraction assessments: fraction NLE, $F(2,38) = 9.69$, $p < 0.001$, fraction comparison, $F(2,38) = 4.27$, $p = 0.021$, fraction arithmetic, $F(2,38) = 8.45$, $p < 0.001$, and fraction

Table 2. Bivariate correlations between all variables.

Variable	1	2	3	4	5	6	7	8	9	10	11	12
1. Units Coordination	-	**−.60**	**.45**	**.52**	**.58**	**.59**	**.34**	**.45**	**−.40**	**.31**	−.09	**.65**
2. Fraction NLE (PAE)	**−.60**	-	**−.66**	**−.55**	**−.55**	**−.46**	−.22	**−.34**	**.49**	**−.32**	.11	**−.59**
3. Fraction Comparison	**.45**	**−.66**	-	**.42**	**.42**	**.39**	.16	**.51**	**−.33**	**.43**	**−.31**	**.48**
4. Fraction Arithmetic	**.52**	**−.55**	**.42**	-	**.54**	**.51**	**.39**	.22	−.21	**.53**	−.11	**.62**
5. Fraction Schemes	**.58**	**−.55**	**.42**	**.54**	-	**.53**	**.35**	**.33**	**−.46**	**.34**	**−.38**	**.70**
6. Raven's Matrices	**.59**	**−.46**	**.39**	**.51**	**.53**	-	**.34**	**.43**	**−.46**	.25	**−.36**	**.66**
7. Working Memory	**.34**	**−.22**	.16	**.39**	**.35**	**.34**	-	−.10	−.20	**.31**	−.10	**.27**
8. Spatial PR	**.45**	**−.34**	**.51**	.22	**.33**	**.43**	−.10	-	−.16	.12	**−.31**	**.44**
9. Whole NLE (PAE)	**−.40**	**.49**	**−.33**	−.21	**−.46**	**−.46**	−.20	−.16	-	**−.36**	.12	**−.35**
10. Arithmetic Fluency	**.31**	**−.32**	**.43**	**.53**	**.34**	.25	**.31**	.12	**−.36**	-	−.17	**.37**
11. Math Anxiety	−.09	.11	**−.31**	−.11	**−.38**	**−.36**	−.10	**−.31**	.12	−.17	-	**−.30**
12. Algebra	**.65**	**−.59**	**.48**	**.62**	**.70**	**.66**	**.27**	**.44**	**−.35**	**.37**	**−.30**	-

Note. PAE = Percent Absolute Error; NLE = Number Line Estimation; PR = Proportional Reasoning. Bolded values are significant at $p < 0.05$.Correlations used pairwise complete observations.

Table 3. Regressions predicting different aspects of fraction knowledge.

Predictor	Fraction NLE β (SE)	Fraction NLE p	Fraction comparison β (SE)	Fraction comparison p	Fraction arithmetic β (SE)	Fraction arithmetic p	Fraction schemes β (SE)	Fraction schemes p
Units Coordination	−0.49 (0.04)	0.005**	0.23 (0.07)	.209	0.36 (0.04)	.028*	0.37 (0.15)	.023*
Arithmetic Fluency	0.18 (0.05)	0.262	0.21 (0.08)	.232	0.27 (0.04)	.079	0.08 (0.17)	.579
Whole Number NLE	0.33 (0.30)	0.026*	−0.07 (0.48)	.659	0.22 (0.25)	.112	−0.27 (0.99)	.056
Math Anxiety	0.00 (0.16)	0.984	−0.06 (0.26)	.676	0.06 (0.13)	.659	−0.23 (0.52)	.078
Spatial Ratio Processing	0.09 (0.12)	0.571	−0.08 (0.19)	.667	−0.13 (.09)	.397	0.17 (0.38)	.265
Working Memory	−0.24 (0.65)	0.130	−0.04 (1.05)	.796	0.23 (.55)	.128	0.14 (2.14)	.341
Ravens	−0.03 (0.21)	0.890	0.19 (0.34)	.366	0.29 (0.18)	.109	−0.06 (0.68)	.739
Age	−0.05 (2.4)	0.697	−0.23 (3.9)	.136	0.03 (2.06)	.813	0.14 (7.98)	
	$F(8, 38) = 4.34$, $p < 0.001$ $R^2 = 47.8\%$		$F(8, 38) = 2.47$, $p = 0.029$ R2 = 34.2%		$F(8, 38) = 5.09$, $p < 0.001$ $R^2 = 51.7\%$		$F(8, 38) = 5.13$, $p < 0.001$ $R^2 = 51.9\%$	

Note. All betas are standardized. NLE = number line estimation.

schemes, $F(2,45) = 6.63$, $p = 0.003$. Post-hoc Bonferroni tests showed that there was a statistically significant difference between students in Stage 1 and Stage 2 on fraction NLE ($p = 0.003$), comparison ($p = 0.032$), and schemes (but not arithmetic); a statistically significant difference between students in Stage 1 and Stage 3 on all four fraction tests (NLE: $p < 0.001$, comparison: $p = 0.034$, arithmetic: $p = 0.009$, schemes: $p < 0.001$); and a statistically significant difference between students in Stage 2 and Stage 3 on fraction arithmetic ($p = 0.031$) and fraction schemes ($p = 0.038$), but not in fraction number line estimation ($p = 0.527$) or fraction comparison ($p = 1.0$). These results are mostly consistent with the bivariate correlations and regression coefficients from the models treating units coordination as a continuous accuracy score.

Relations Between Units Coordination and Algebra Knowledge

As shown in Table 2, units coordination scores were also strongly correlated with algebra knowledge ($r = 0.65$, $p < 0.001$). We used standardized regressions to isolate the specific association between units coordination and algebra knowledge. We regressed algebra knowledge on units coordination, controlling for whole number estimation and arithmetic fluency, working memory, spatial ratio processing, Ravens Matrices, math anxiety, and age. Even with all controls, units coordination explained significant variance in algebra performance, $\beta = 0.42$, $p = 0.005$. Age ($\beta = 0.24$, $p = 0.047$) and Ravens Matrices ($\beta = 0.34$, $p = 0.042$) also significantly predicted algebra knowledge.

Units coordination stages also explained significant variance on the overall algebra assessment in an ANCOVA controlling for the same covariates, $F(2,37) = 16.63$, $p < 0.001$. Follow-up pairwise Bonferroni contrasts showed that algebra scores were lowest among children at Stage 1 of units coordination ($M = 26.4\%$, $SD = 9.1$), who showed a statistically significant difference from students at Stage 2 units coordination ($M = 55.8\%$, $SD = 16.2$). Children at Stage 3 of units coordination ($M = 70.8\%$, $SD = 13.1$) performed statistically significantly better in algebra than children at the other two stages.

Predictors of Overall Algebra Knowledge

To address our final research question, we conducted a standardized linear regression predicting algebra knowledge from all other variables, to test which of the aspects of fractions knowledge contributed unique variance to algebra knowledge when controlling for the other aspects of fractions knowledge and related skills. We conducted this regression in two steps: In Model 1, we entered only fraction predictors and units coordination, to isolate the unique variance explained by each fraction measure when controlling for the other fraction measures and units coordination. Then, in Model 2, we entered all predictors including all covariates. Results of both models are shown in Table 4.

Table 4. Regression predicting algebra knowledge.

	Model 1 (Fractions and UC only)			Model 2 (All controls)		
Predictor	*β*	*SE*	*p*	*β*	*SE*	*p*
Fraction Schemes	0.394	0.119	.004**	0.359	0.138	.024*
Fraction Arithmetic	0.194	0.436	.133	0.073	0.546	.650
Fraction Comparison	0.042	0.276	.750	0.044	0.296	.759
Fraction NLE	−0.017	0.465	.912	−0.061	0.506	.721
Units Coordination	0.305	0.114	.028*	0.237	0.129	.126
Arithmetic Fluency				0.099	0.154	.499
Whole Number NLE				0.091	0.921	.531
Math Anxiety				0.066	0.414	.560
Spatial Ratio Processing				0.002	0.307	.990
Working Memory				0.008	1.712	.953
Ravens				0.337	0.553	.045*
Age				0.197	6.309	.093
	$F(5, 40) = 12.9$, $p < 0.001$ $R^2 = 61.8\%$			$F(12, 33) = 6.66$, $p < 0.001$ $R^2 = 70.8\%$		

Note. All betas are standardized. NLE = Number line estimation.

Before interpreting the results, we checked the Variance Inflation Factor (VIF) of all predictors in both models, given the correlations observed between our predictor variables. In Model 1, the VIF for all predictors was lower than 3, and in Model 2, the highest VIF level was fraction number line estimation which had a VIF of 3.23. These results did not indicate any unacceptable collinearity (James et al., 2023).

In Model 1, both fraction schemes ($\beta = 0.39$, $p = 0.004$) and units coordination ($\beta = 0.31$, $p = 0.028$) significantly predicted algebra knowledge, $F(5, 40) = 12.94$, $p < 0.001$, $R^2 = .618$. None of the other fraction measures were significant predictors of algebra knowledge. In the full model (i.e., Model 2), only fraction schemes continued to significantly predict algebra performance ($\beta = 0.36$, $p= 0.024$) after accounting for all the other aspects of fraction knowledge, units coordination, and all the covariates, $F(12, 33) = 6.67$, $p < 0.001$, $R^2 = .708$. The effect was large; scoring one standard deviation better on the fraction schemes assessment was associated with an improvement of 0.36 standard deviations on the algebra test, even after accounting for the influence of all other variables. Importantly, fraction NLE ($\beta = -0.06$, $p = 0.721$), comparison ($\beta = 0.04$, $p = 0.759$), and arithmetic ($\beta = 0.07$, $p = 0.650$) were not significantly associated with algebra in the full model. Units coordination was no longer a statistically significant predictor in Model 2 after accounting for the math and general cognitive skills ($\beta = 0.24$, $p = 0.126$). Among the covariates, only students' performance on the Raven's Progressive Matrices test of nonverbal reasoning was a statistically significant predictor of algebra knowledge after accounting for all other predictors ($\beta = 0.34$, $p = 0.04$).

Discussion

Our study assessed eighth graders using common measures of fractions knowledge from psychology (i.e., NLE, comparison, and arithmetic) alongside fraction schemes and units coordination, as well as a host of other relevant math and cognitive skills, to offer new evidence about the fractions-algebra connection. Although quantitative fractions-algebra studies have privileged measures of fraction magnitude and arithmetic, we found that units coordination and fraction schemes, but *not* fraction magnitude or arithmetic, explained significant variance in algebra knowledge when the fractions predictors were all included in the same model. Only fraction schemes, which are rarely discussed or measured in psychological studies, continued to predict algebra knowledge even with strong math and domain-general controls. These findings are striking, especially given the dominance of fraction magnitude in quantitative fractions-algebra studies (e.g., Booth et al., 2014; Hurst & Cordes, 2018a; Mou et al., 2016) and in psychological fractions research more generally (e.g., Bailey et al., 2017; Faulkenberry & Pierce, 2011; Matthews & Chesney, 2015; Resnick et al., 2016; Torbeyns et al., 2015).

Our findings also extend prior research showing a relation between units coordination and fraction knowledge. Even after statistically controlling for related math and cognitive skills, students' units coordination scores were still uniquely associated with their fraction schemes, arithmetic, and NLE performance, but not with magnitude comparison. In the models treating units coordination ability as a categorical stage, all four aspects of fraction knowledge were statistically significantly associated with units coordination.

Making Sense of Relations Among Aspects of Fractions Knowledge and Units Coordination

Our results show that units coordination and fraction schemes, which have generally not been included in fractions research from the field of psychology, are closely related to some aspects of fractions knowledge that are privileged by psychologists, including fraction NLE and arithmetic. These close associations are not surprising. Steffe's (2002) reorganization hypothesis argues that students' fraction knowledge derives from their ways of operating with whole numbers. Students who have developed the understanding that numbers are nested within other numbers (i.e., are operating with a tacitly nested number sequence, Ulrich, 2015) can then reverse operations to see a fraction like ⅓ as the result of partitioning a whole unit of 1 into three equal parts. Seeing units in this reversible way allows students to view fractions as measures, such as seeing ⅔ as two units of ⅓ (Wilkins et al., 2021). Viewing fractions in this way – as measures or magnitudes – would support students' success on NLE tasks.

Constructivist theory about fraction schemes posits that students use these schemes to organize their interpretations of and interactions with fractions in all contexts, including comparison, NLE, and arithmetic (e.g., Norton & Hackenberg, 2010; Steffe & Olive, 2010; Wilkins & Norton, 2018), and many researchers suggest that units coordination is foundational for fraction reasoning (e.g., Boyce & Norton, 2016; Empson et al., 2006; Hackenberg, 2007, 2010; Hackenberg & Lee, 2015; Hackenberg & Tillema, 2009; Norton & Wilkins, 2010). Our results are consistent with Behr et al. (1983) theory and with various hypothetical learning trajectories for fractions (e.g., Norton & Wilkins, 2012) that suggest that units coordination and fraction schemes are prerequisites for building other aspects of fractions knowledge (e.g., fraction as measure or magnitude, fraction arithmetic, etc.), but future research is needed to examine how the relations between different aspects of fractions develop over time.

Our results also highlight the importance of measurement choices in fractions research. Fraction NLE and comparison are typically considered to measure the same construct – fraction magnitude knowledge – but in our results fraction NLE was strongly correlated with both units coordination and fraction schemes whereas fraction comparison was only moderately correlated with these measures. Fraction NLE, but not comparison, was associated with units coordination even after controlling for other math and domain-general skills. Perhaps NLE is more closely related to fraction schemes and units coordination because the tasks all rely on visuospatial processing (Simms et al., 2016) and can be addressed using partitioning strategies (Zhang et al., 2017). Fraction comparison tasks, on the other hand, lack a visuospatial element, and students can solve them using many strategies, some of which attend to whole number relations rather than holistic fraction magnitudes (Fazio et al., 2016). The speeded version of this task, used in this study, is designed to elicit an approximate, analog sense of fraction magnitudes (e.g., Matthews & Lewis, 2017; Schneider & Siegler, 2010). It is possible an unspeeded version of this task would more strongly correlate with units coordination. An alternative explanation, which further research could test, is that the fraction comparison problems only included fractions less than one, whereas the fraction schemes, units coordination, and NLE tests all included improper fractions or mixed numbers.

Fraction Schemes and Units Coordination May Drive the Fractions-Algebra Connection

Our findings provide the first evidence that students' fraction schemes and units coordination have *unique* relations with algebra knowledge, over and above the shared influence of general math skills and domain-general cognitive skills. Prior qualitative studies showed that students who have difficulty reasoning with units coordination and fraction schemes also have difficulty reasoning algebraically (e.g., Hackenberg, 2010; Hackenberg & Lee, 2015), and our results suggest that this association is not likely to be explained by other aspects of students' general math knowledge or broader cognitive skills. Cognitive skills like working memory are intertwined with reasoning about units coordination, fractions, and algebra (e.g., Jordan et al., 2013; Peng et al., 2016; Ünal et al., 2022). For example, Norton et al. (2023) showed that preservice teachers relied on working memory in solving fraction multiplication problems; however, those preservice teachers with more advanced fraction schemes were able to use units coordination to group together multiple steps and free up working memory resources, which facilitated problem-solving. Our results show that even when we statistically account for individual differences in working memory and many other cognitive skills, the relation between 8th graders' fraction schemes and algebra performance persists.

Importantly, our results suggest that the fraction schemes-algebra relation is stronger than the relation of either fraction magnitude knowledge or fraction arithmetic skill with algebra knowledge, despite many studies showing that fractions magnitude and arithmetic uniquely predict algebra performance (e.g., Barbieri et al., 2021; Booth & Newton, 2012; Booth et al., 2014; DeWolf et al., 2015; Hurst & Cordes, 2018a, 2018b). The dominance of fraction schemes is not likely to be an artifact of the test capturing more variance, as we also observed wide distributions of scores on the other fraction tests. Nor does it seem to be an artifact of the inclusion of improper fractions in fraction schemes, because the NLE tasks also included improper fractions. Why were students' schemes scores so closely related to algebra, whereas their magnitude and arithmetic scores were less so? We consider two plausible explanations.

First, theoretical descriptions of fraction schemes drawn from evidence from teaching experiments suggest that fraction schemes are more overarching and foundational than any one aspect of fractions (e.g., Hackenberg, 2007, 2013; Norton & Hackenberg, 2010; Steffe, 2002; Steffe & Olive, 2010; Wilkins & Norton, 2018). That is, the fraction schemes assessment may be capturing students' underlying mental models that they use to interpret and interact with fractions in any context, including when comparing fractions, estimating their position on a number line, or solving arithmetic problems. If this is true, then current theory around fraction magnitude and arithmetic knowledge may need to be revisited to consider whether these constructs might be reframed as a subset or a result of fraction schemes.

However, some of our findings are inconsistent with this explanation. Some students had strong fraction arithmetic, NLE, or comparison performance despite having relatively weak fraction schemes performance. Further, as noted above, fraction arithmetic and magnitude tasks can be solved using many strategies, not all of which rely directly upon partitioning, iterating, disembedding, or units coordination (e.g., Kalra et al., 2020). For example, strategy reports not included in this analysis showed that students sometimes used strategies based on whole number reasoning, such as deciding that 3/4 is bigger than 3/5 because if the numerators are the same, the fraction with the smaller denominator is greater.

A second explanation for the dominance of fraction schemes in explaining algebra performance is that the fraction schemes test may be capturing students' *relational* understanding of fractions, which has been shown to be associated with algebra knowledge (DeWolf et al., 2015). The fraction schemes assessment requires an understanding of the relational structure between numerator and denominator, as well as between unit fractions, non-unit fractional parts, and wholes. For example, students need to recognize that the whole candy bar should be longer than the original bar on the item "The stick shown below is 4/5 as long as a whole candy bar. Draw the whole candy bar." They also need to recognize that if the original bar is cut into four pieces, each piece is 1/5 of the whole. Students who can keep track of these complex quantitative relations simultaneously may use similar relational reasoning when solving complex algebraic problems like "If $x + y = 12$ and $2x + 5y = 36$, what are the values of x and y?". On the other hand, children may have answered the fraction comparison, NLE, and arithmetic items using procedures or whole number reasoning that would not necessarily require relational reasoning.

Limitations

Although our sample was somewhat diverse geographically, most students were White and relatively high-income. Students (or their parents/guardians) also had to self-select into a study about children's mathematical thinking, which may have biased our sample to include mostly students with relatively high math achievement and/or math interest. Our future work will aim to recruit a broader sample that includes students of varying demographics and ability levels. It will also be important for future work to include larger sample sizes to provide statistical power to detect smaller effects and take advantage of more sophisticated statistical techniques like factor analysis.

A major strength of our design was the inclusion of many math and broader cognitive assessments as covariates, which allowed us to examine relations among fractions knowledge, units coordination, and algebra controlling for those related skills. However, we did not include a measure of children's vocabulary or reading comprehension. Especially because the fraction schemes, units coordination, and algebra tests involved more written language than the other fraction assessments, future studies should include a reading assessment to be able to isolate the fraction-specific relation with algebra, over and above any shared influence of reading skills.

Future Directions

In the analyses presented here, we parsed fractions knowledge into distinct components but considered algebra knowledge as a single outcome measure. However, algebra knowledge is also a broad and multifaceted construct. Some have parsed algebra into "big ideas" or content areas: generalized arithmetic, functional thinking, and equivalence (Blanton et al., 2018). Alternatively, Kieran (2007) defined school algebra as consisting of three types of activities: generational, transformational, and global. Star and Rittle-Johnson (2008) propose parsing algebra knowledge into conceptual knowledge, procedural knowledge, and flexibility. It is likely that different aspects of fractions knowledge relate to different aspects of algebra knowledge, and our ongoing work will directly investigate these more nuanced relations.

Our study showed that fraction schemes are strongly associated with algebra knowledge, even when controlling for other skills with fractions, overall math skills, and general cognition, but many questions remain. Longitudinal studies could illuminate when and how specific aspects of fractions knowledge and algebra knowledge co-develop. A few longitudinal studies have shown that fractions knowledge predicts algebra knowledge, but they have not included fraction schemes (Barbieri et al., 2021; Booth et al., 2014; Liang et al., 2018; Siegler et al., 2012). A longitudinal approach would also be helpful to capture how the progressive construction of units coordination from level 1 to 2 to 3 relates to students' learning in different aspects of fractions and algebra over time.

We focused on more advanced fraction schemes and operations (i.e., the splitting operation, the reversible partitive fraction scheme, and the iterative fraction scheme) in part due to the age of our sample, but future work should seek to measure the developmental progression of fraction schemes more fully. Researchers have shown that children tend to progress in a predictable order of fraction schemes including pre-fraction, part-whole fraction scheme, partitive fraction scheme, reversible partitive fraction scheme, and iterative fraction scheme (Steffe & Olive, 2010). Alternatively, Wilkins and Norton (2018) frame the development of fraction schemes in terms of measurement, progressing from the measurement scheme for unit fractions, to the measurement scheme for proper fractions, and finally to the general measurement scheme for fractions.

As we have argued throughout the discussion, psychologists have much to gain from leveraging the important constructs of units coordination and fraction schemes generated by math education researchers. However, we believe that the benefits of such boundary-spanning work is likely to be bi-directional. For example, our results show some evidence that there is important variability both between and within stages, as shown in Figures 4 and 5. Future work could leverage psychometric tools like factor analysis, construct modeling, Rasch analyses, and Item-Response Theory (IRT) to develop assessments that might be used to test the hypothesized stagewise development of units coordination and fraction schemes. Work like this has been done in other domains like equal sign knowledge (e.g., Rittle-Johnson et al., 2011; Fyfe et al., 2018), teacher's mathematical knowledge for teaching (e.g., Hill et al., 2004; Hill et al., 2008), and different early mathematics learning trajectories (Clements, et al., 2008; Kutaka et al., 2023).

Lastly, future work should examine how instructional and socioemotional factors influence the relations between fractions and algebra knowledge. Our sample was entirely drawn from the US, which tends to emphasize part-whole interpretations and area model visualizations of fractions. The relations between fractions and algebra knowledge may differ in educational systems that emphasize measurement interpretations or number line models (e.g., Resnick et al., 2023; Torbeyns et al., 2015). Future work is also needed to unpack the relations among fractions knowledge, algebra knowledge, and socioemotional factors such as math anxiety, math identity (e.g., Miller-Cotto & Lewis, 2020) and motivation (e.g., Wang et al., 2015).

Conclusion

Our findings offer a powerful illustration of how a synthesis of methods from different research traditions stands to contribute to our understanding of mathematical development. First, we found that two constructs whose study has heretofore been confined to mathematics education research, fraction schemes and units coordination, held more explanatory power for our outcomes of interest than fractions magnitude did. These findings demonstrate that there are compelling reasons to update dominant psychological theories and to interrogate the privileged role that fractions magnitude plays in their conception of fractions knowledge. More generally, we suspect that this is but one case in which breaking silos and integrating the theory and methods from separate traditions can enhance the power of research in mathematical thinking and learning. We believe that cultivating a healthy respect for the contributions of different methods and perspectives can yield interdisciplinary collaborations and products that are meaningful for both research and practice.

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