Measuring and Supporting Proportional Reasoning:

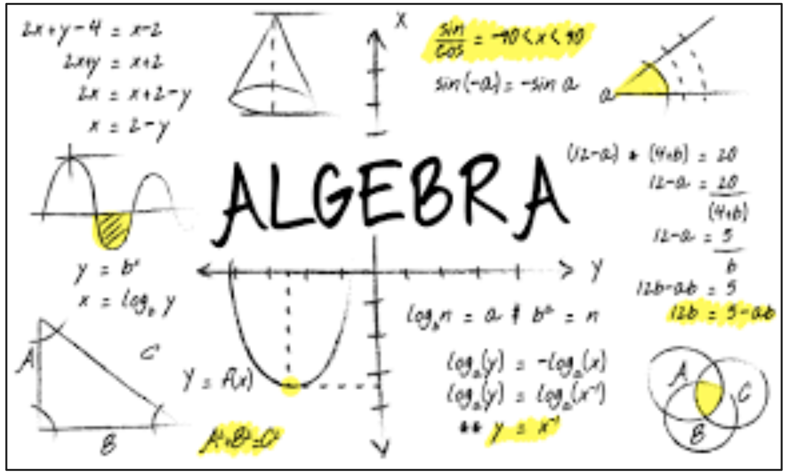
An Interdisciplinary Approach

Dissertation Defense

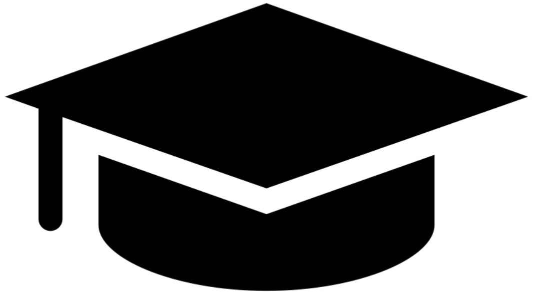
June 24, 2022

Alexandria Viegut, Dept. of Educational Psychology, UW – Madison

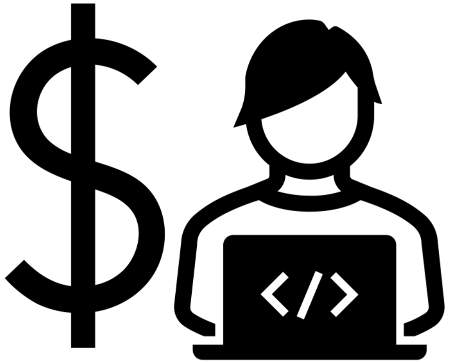
Algebra is crucial for students' outcomes



Gamoran & Mare, 1989



Adelman, 2006
Chen, 2013



Moses & Cobb, 2002

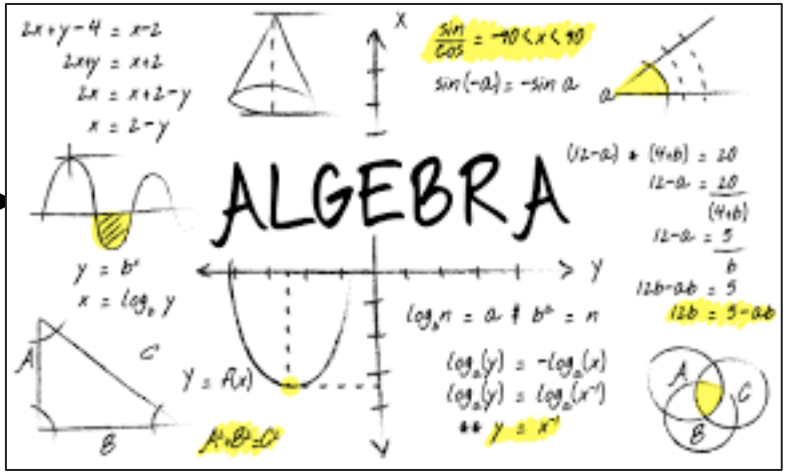
Fractions may be the keys to the gate

- Fractions scores at age 10 predict algebra scores 5-6 years later, even controlling for other math skills, IQ, reading, etc. (Siegler et al., 2012)
- Correlated in many age groups (Hurst & Cordes, 2018; Powell et al., 2019)
- Fractions predict how much students learn from algebra instruction (Booth et al., 2014)

Fractions may be the keys to the gate

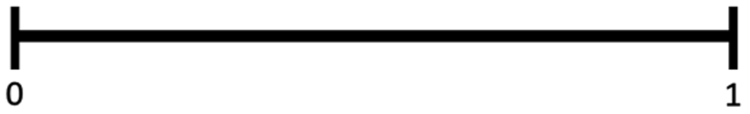
Fractions Knowledge

Why?

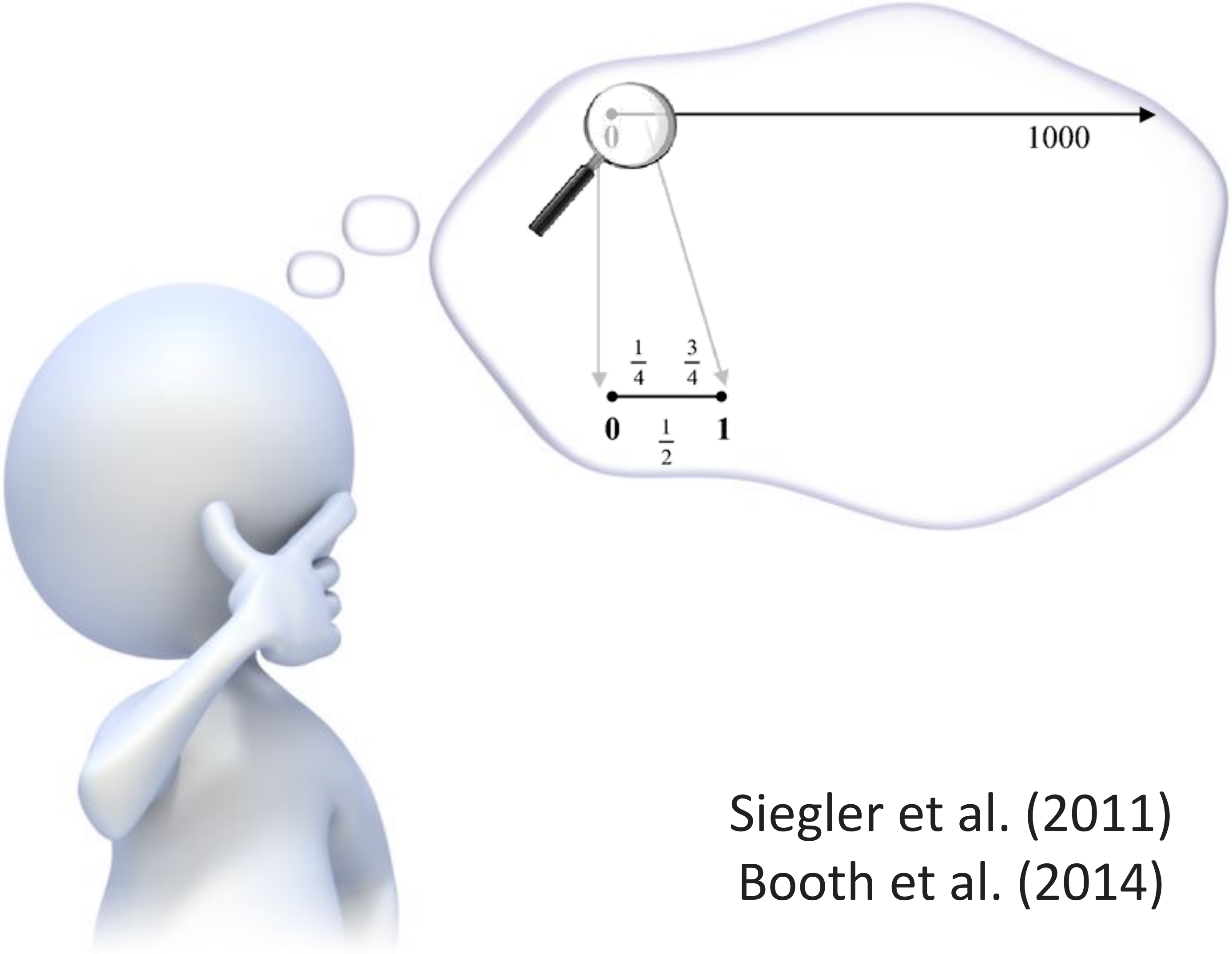


Some research suggests it is all about *magnitude*

$$\frac{3}{5}$$



$$\frac{7}{8} \quad \frac{3}{5}$$

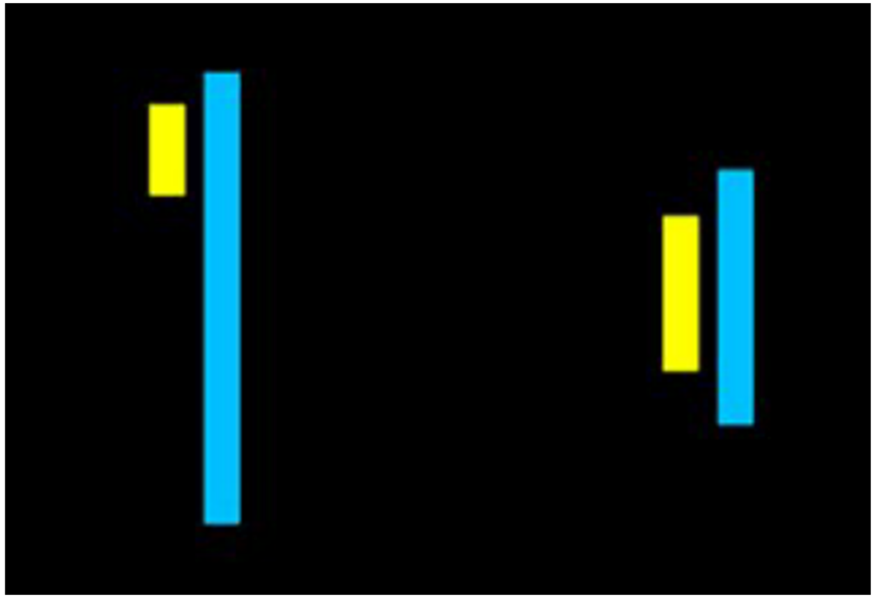


Siegler et al. (2011)
Booth et al. (2014)

Some research suggests it is all about *magnitude*

$$\frac{5}{9} \overset{?}{>} \frac{2}{3}$$

Having an approximate sense of fraction magnitudes helps people avoid common errors!



We have intuitions about visual proportions that can help us with this “fraction sense”?

But other research shows that fractions knowledge is much *broader*

Fractions Knowledge

Subconstructs
- Part-Whole
- Quotient
- Ratio
- Operator
- Measure

Magnitude → **Measure**

Schemes
- Partitive
- Iterative
- Reversible

Operations
- Splitting
- Disembedding
- Units Coordination

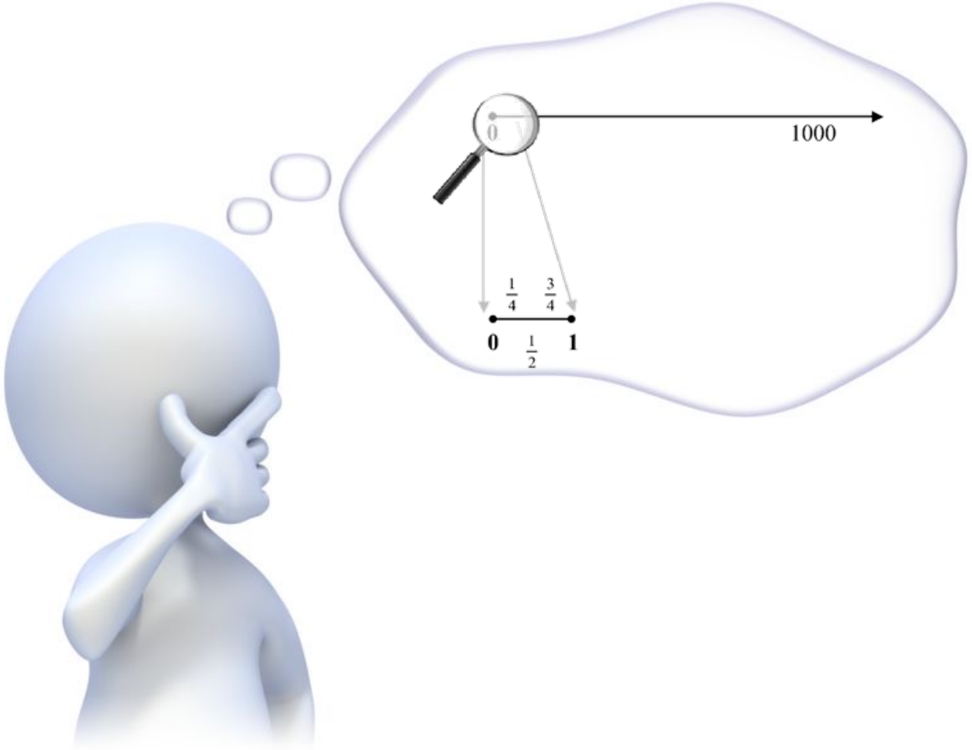
Models
- Area Model
- Discrete Set
- Number Line
- Continuous Proportion
- Symbolic

Tasks
- Estimate
- Compare
- Transform
- Arithmetic
- Justify & Explain
- Construct

Development →

Study 1 focuses on *magnitude*.

What kind of lesson with number lines helps 3rd-4th graders build an approximate “fraction sense”?



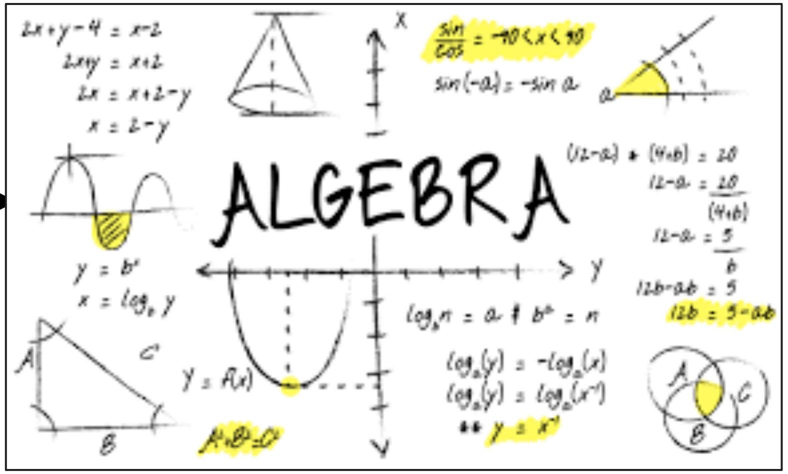
Study 2 focuses on *broader* fractions knowledge

Fractions Knowledge

Common in psych:
Magnitude
Arithmetic

Common in math ed:
Schemes
Units Coordination

Why? →

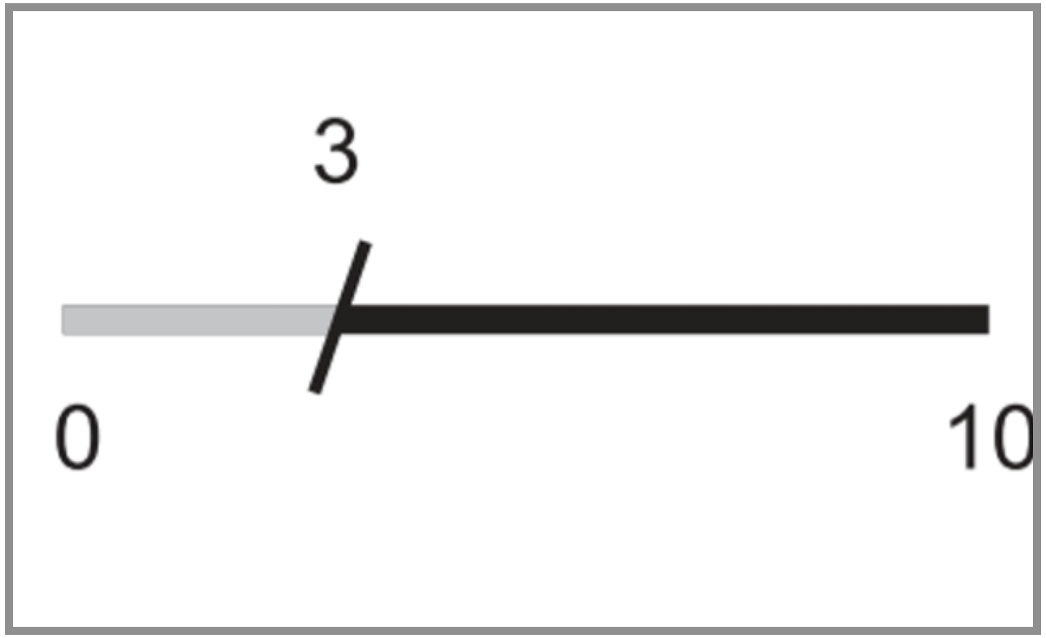


Key Questions of this dissertation

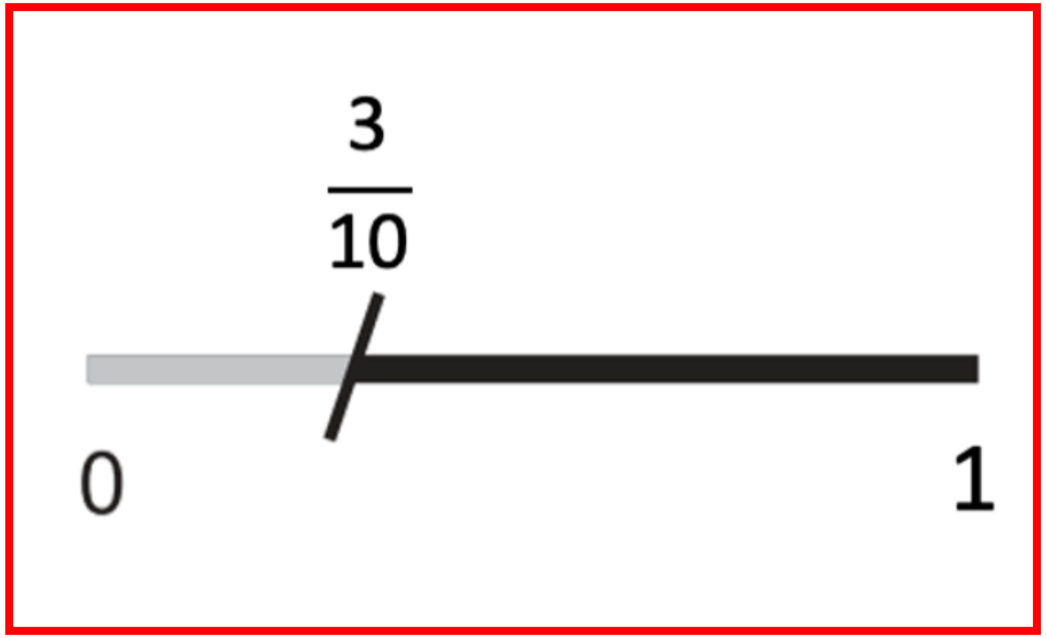
Study 1:
How can we best use **number lines** to support students' fraction **magnitude** knowledge?

Study 2:
What can we learn about **why fractions relate to algebra** from including **multiple measures** of fractions knowledge?

Study 1: How can we use number lines to support students' fraction magnitude knowledge?



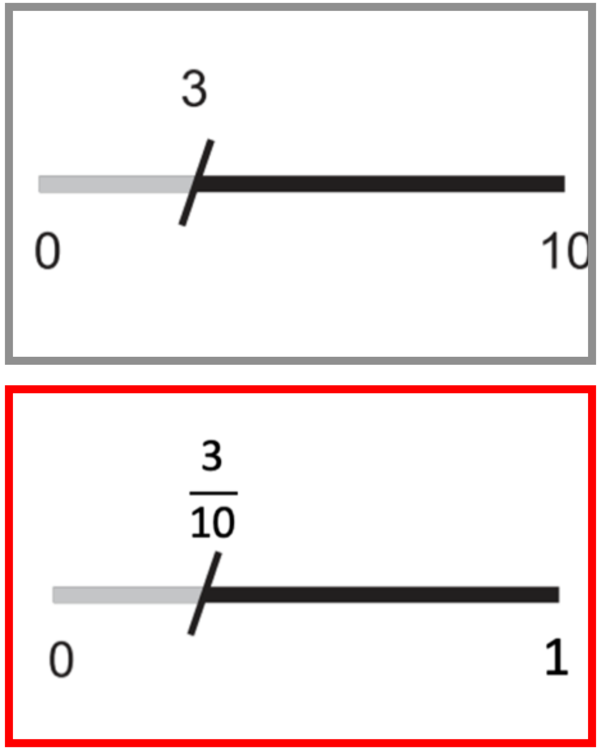
How big is 3 compared to 10?



How big is 3/10 compared to 1?

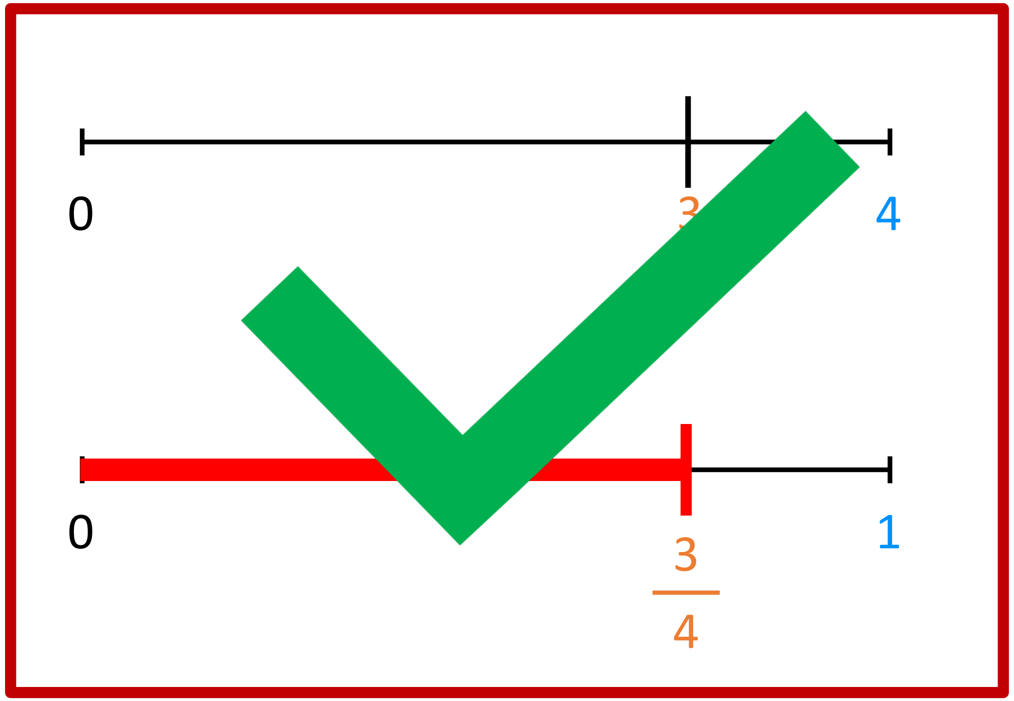
Knowing how whole number sizes relate might help kids estimate fractions.

Study 1: How can we use number lines to support students' fraction magnitude knowledge?

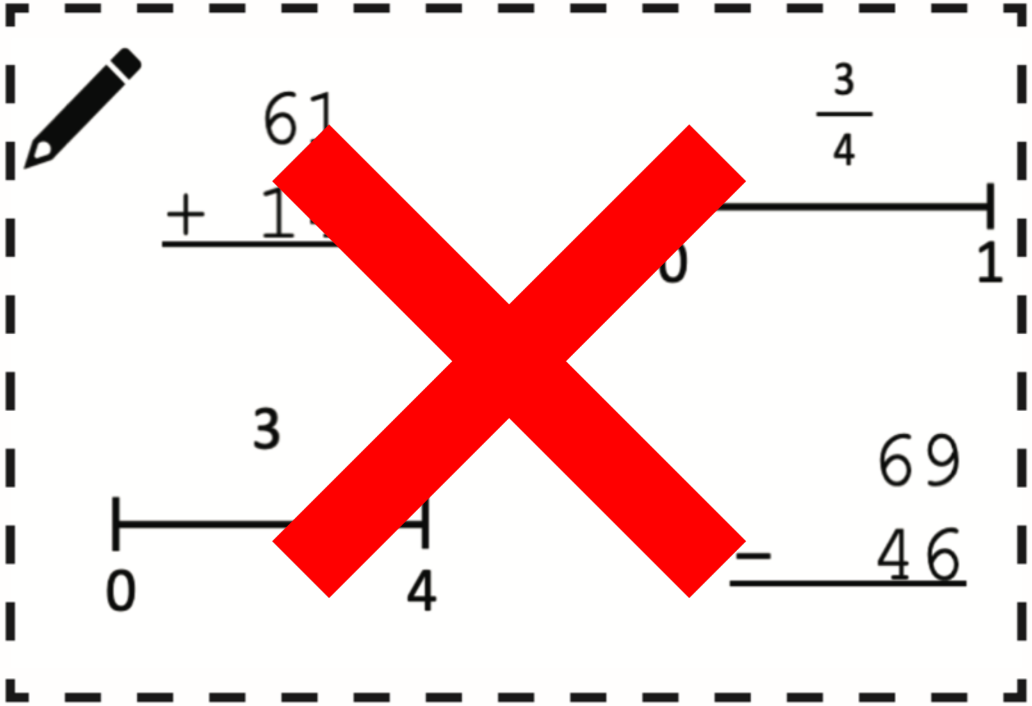


Knowing how whole number sizes relate might help kids estimate fractions.

In a prior study with 2nd – 3rd graders:

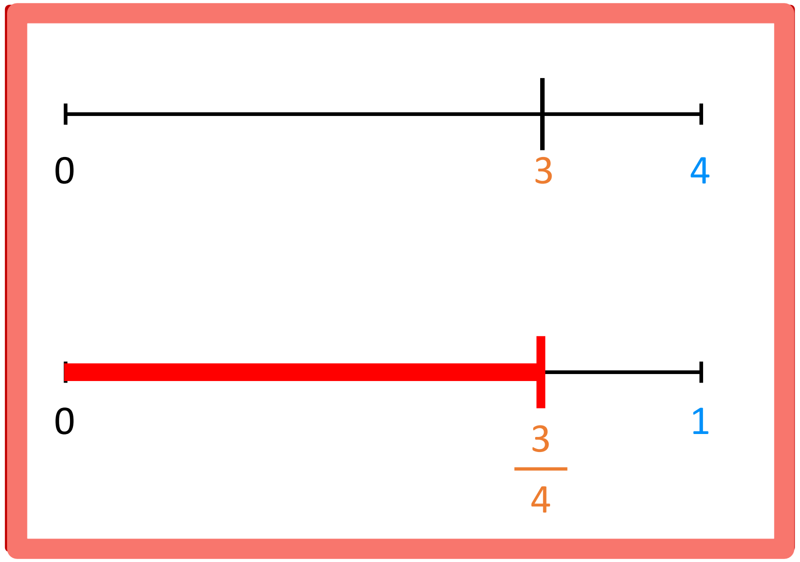


Analogy Lesson

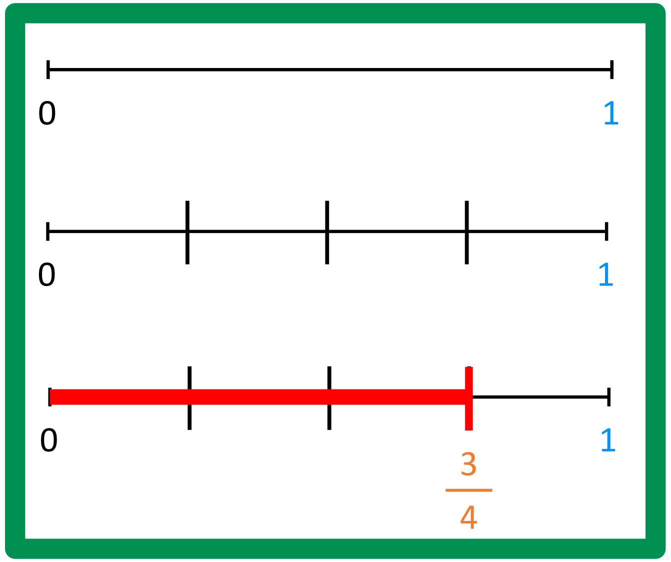


No Lesson

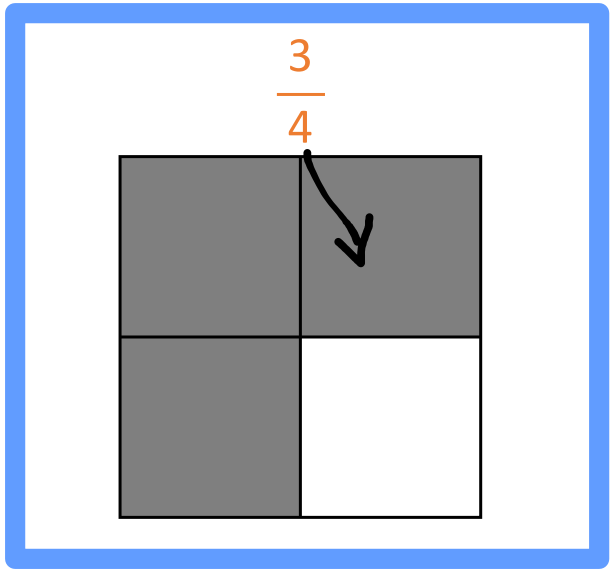
Study 1: Which lesson helps 3rd-4th graders (N = 86) learn to estimate and compare fractions?



vs



vs



Analogy

If I know how big 3 is compared to 4, I know how big ¾ is!

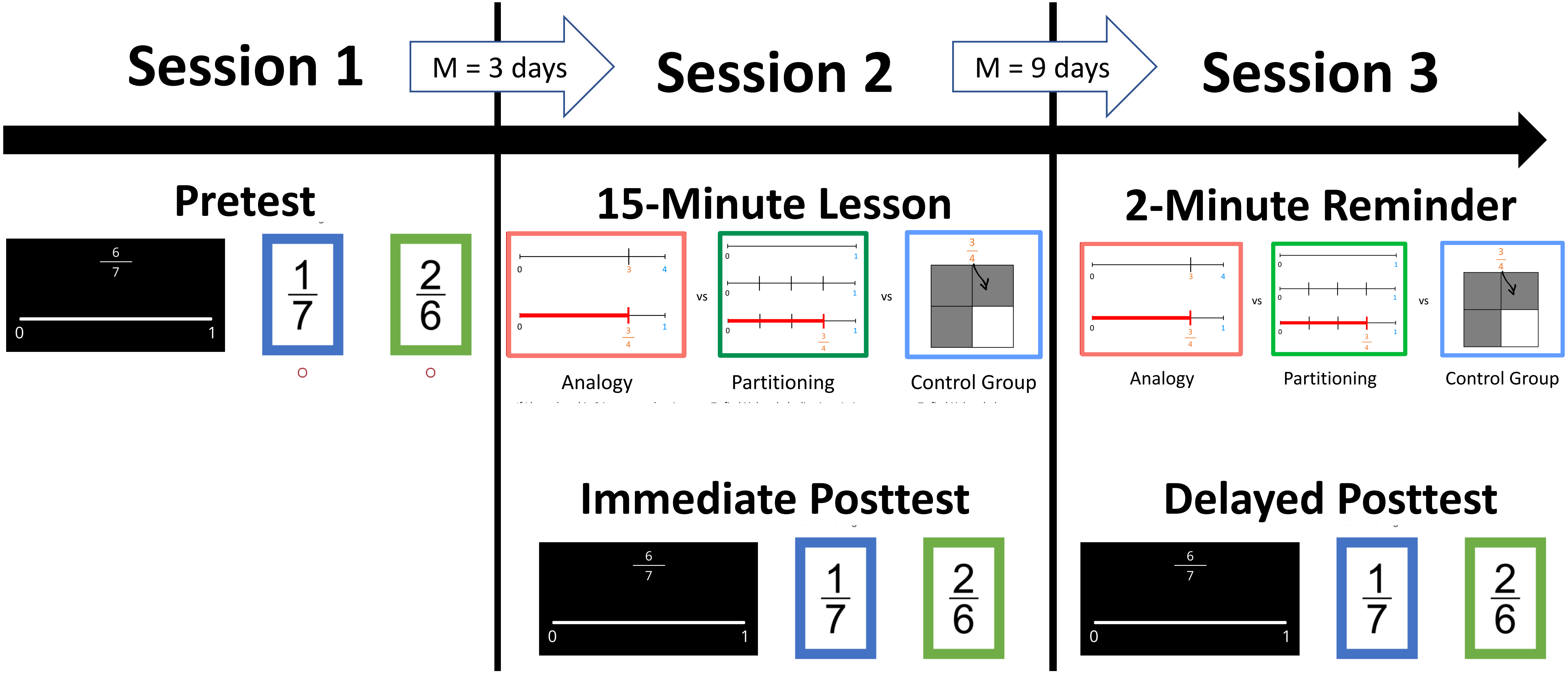
Partitioning

To find ¾, break the line into 4 pieces, and count over 3.

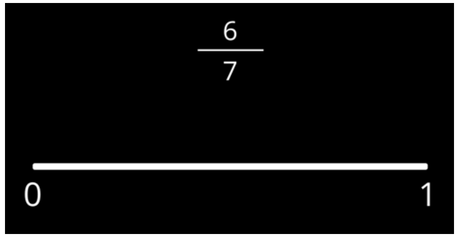
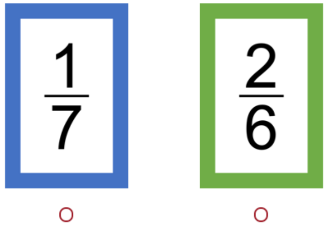
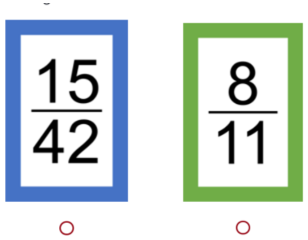
Control Group

To find ¾, break the square into 4 pieces, and shade in 3.

Study 1 Design

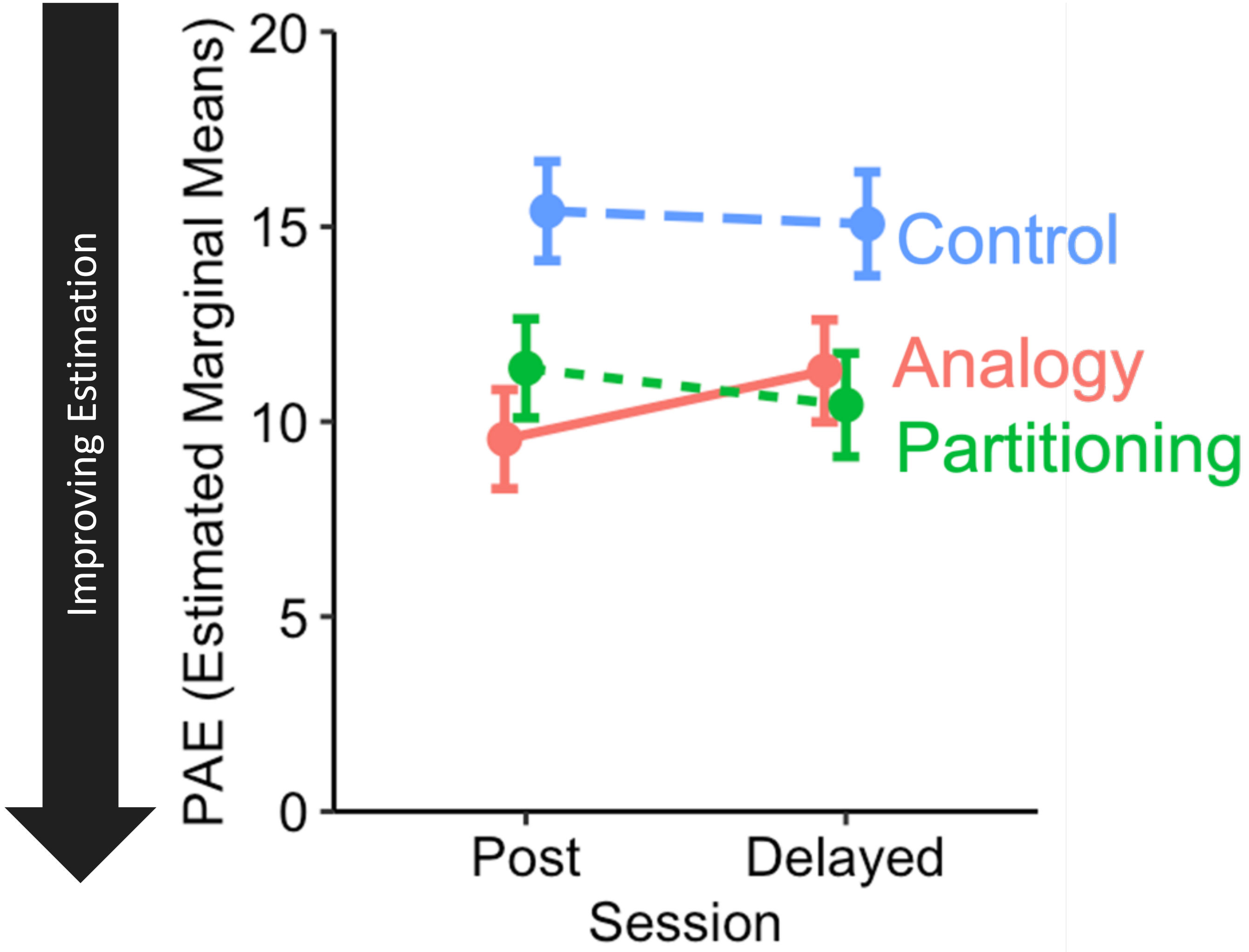


Study 1 Hypotheses

	Outcome	Image	Hypothesis
H1	Number Line Estimation (Learning)		Analogy & Partitioning > Control
H2	Comparison (Transfer)		Analogy & Partitioning > Control
H3	Estimation & Comparison with Large-Denominator Fractions		Analogy > Partitioning & Control

Q1. Did children learn to *estimate* fractions?

H1. Analogy and Partitioning will have lower error than Control



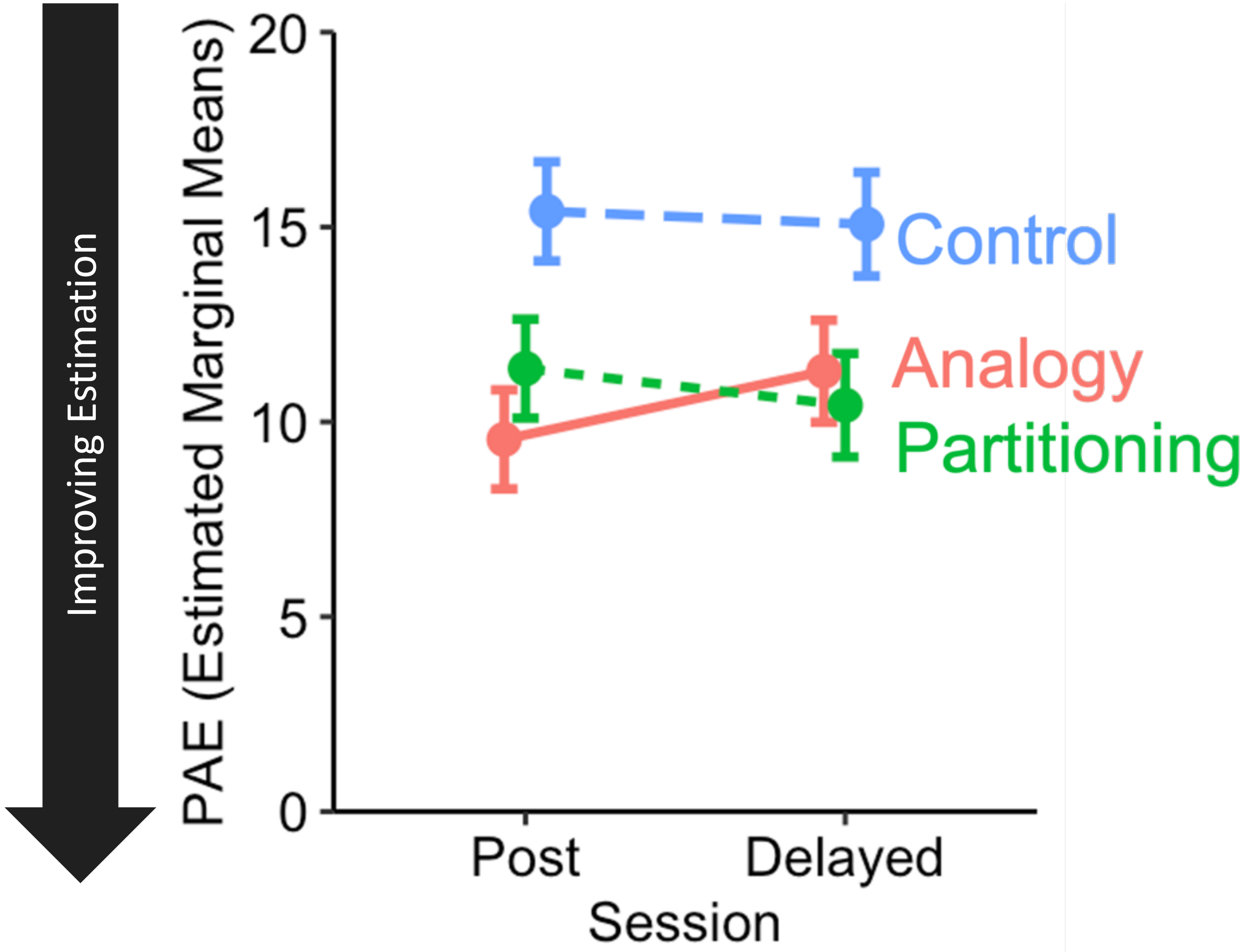
As hypothesized, both Analogy & Partitioning groups had lower PAE than Control group at *Immediate* Posttest.

Post ~ Pre + Condition + WhNumNLE

	$\beta_{standardized}$ (SE)	*p*
Intercept	0.000 (1.9)	.207
Pretest	0.41 (.08)	<.001***
Condition: Analogy	-0.28 (1.8)	.002**
Condition: Partitioning	-0.20 (1.8)	.027*
Whole Number PAE	0.40 (.18)	<.001***

Q1. Did children learn to *estimate* fractions?

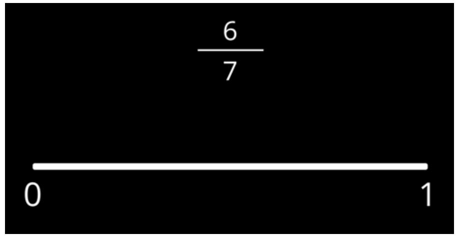
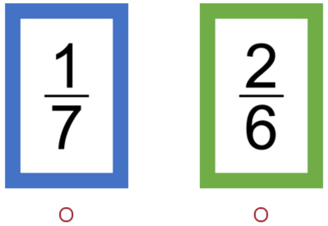
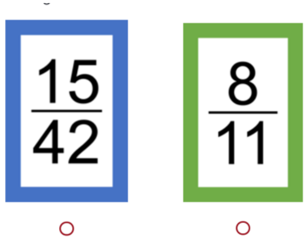
H1. Analogy and Partitioning will have lower error than Control



At *Delayed Posttest*, both Analogy & Partitioning groups sustained lower PAE than Control group.

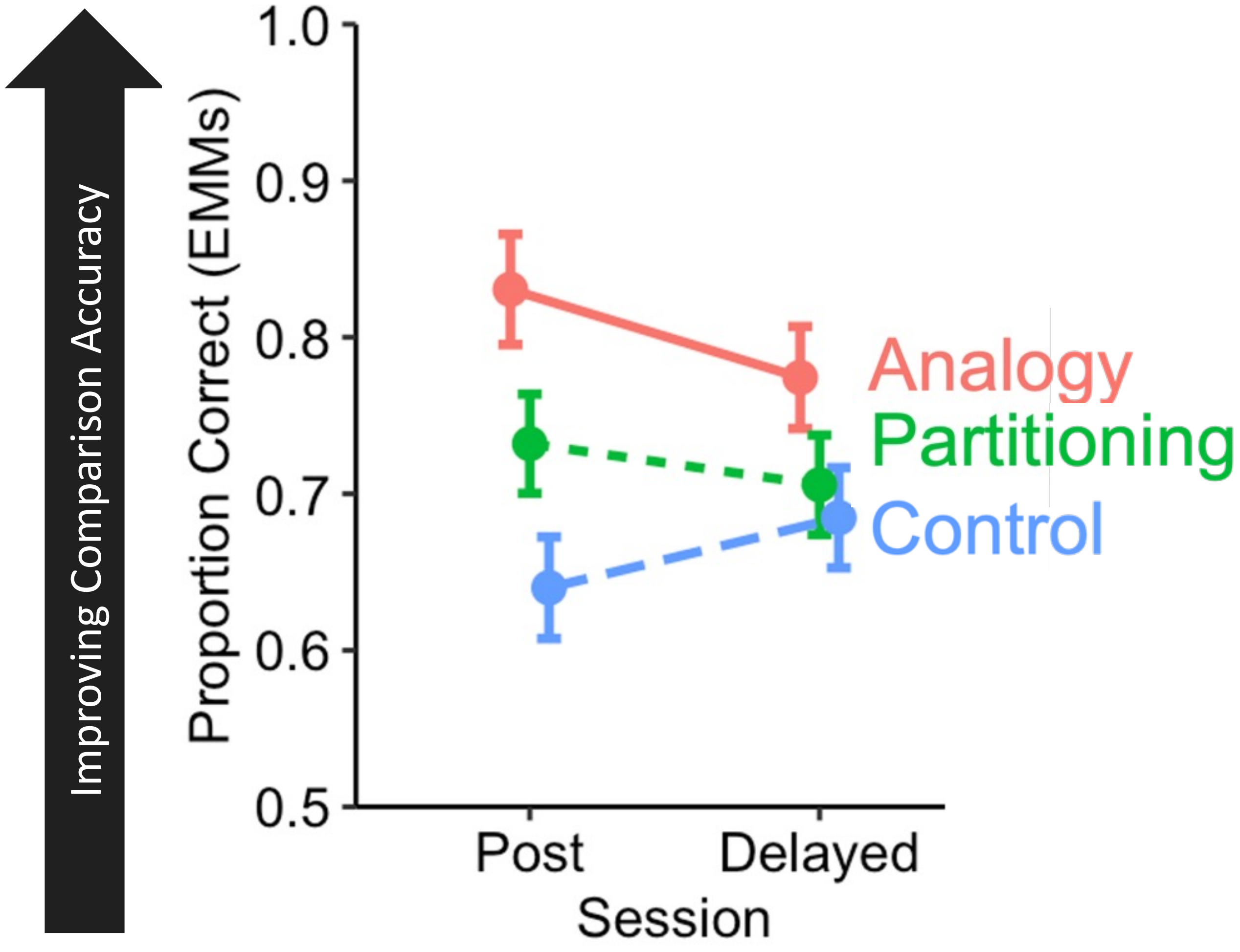
(when controlling for pretest and whole number estimation)

A1. Both Analogy & Partitioning groups had lower error on Number Line Estimation than Control

	Outcome	Image	Hypothesis
H1	Number Line Estimation (Learning)		Analogy & Partitioning > Control
H2	Comparison (Transfer)		Analogy & Partitioning > Control
H3	Estimation & Comparison with Large-Denominator Fractions		Analogy > Partitioning & Control

Q2. Did children learn to *compare* fractions?

H2. Analogy and Partitioning will have higher accuracy than Control



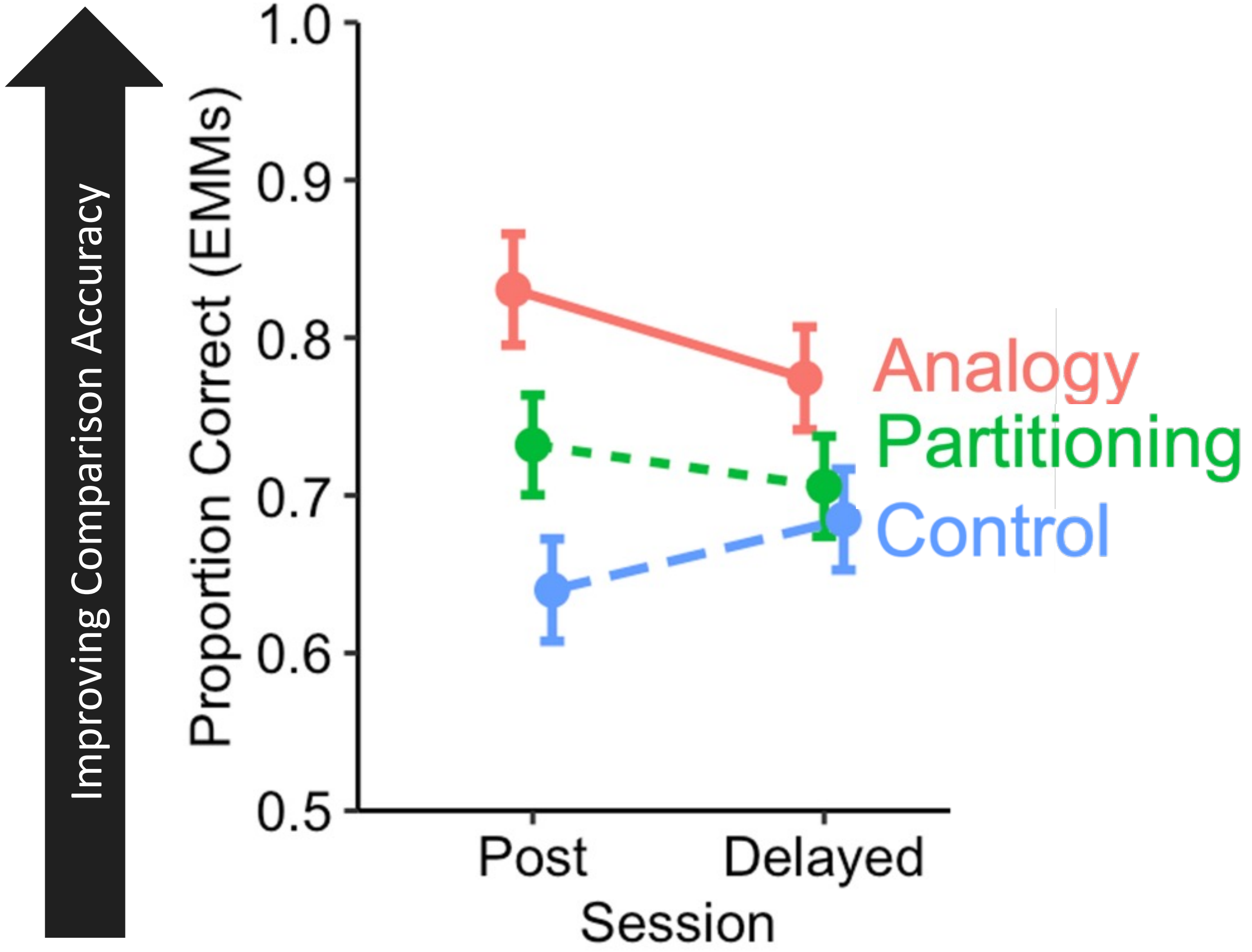
As hypothesized, both Analogy & Partitioning groups had higher accuracy than Control group at *Immediate* Posttest.

Post ~ Pre + Condition + Pre*Condition

	$\beta_{standardized}$ (SE)	p
Intercept	0.00 (.09)	.422
Pretest	1.06 (.12)	<.001***
Pretest × Analogy	-0.81 (.17)	<.001***
Pretest × Partitioning	-0.37 (.17)	.115
Condition: Analogy	1.12 (.13)	<.001***
Condition: Partitioning	0.56 (.12)	.022*

Q2. Did children learn to *compare* fractions?

H2. Analogy and Partitioning will have higher accuracy than Control



At *Delayed Posttest*, **only the Analogy group** performed better than the Control group

(when controlling for pretest)

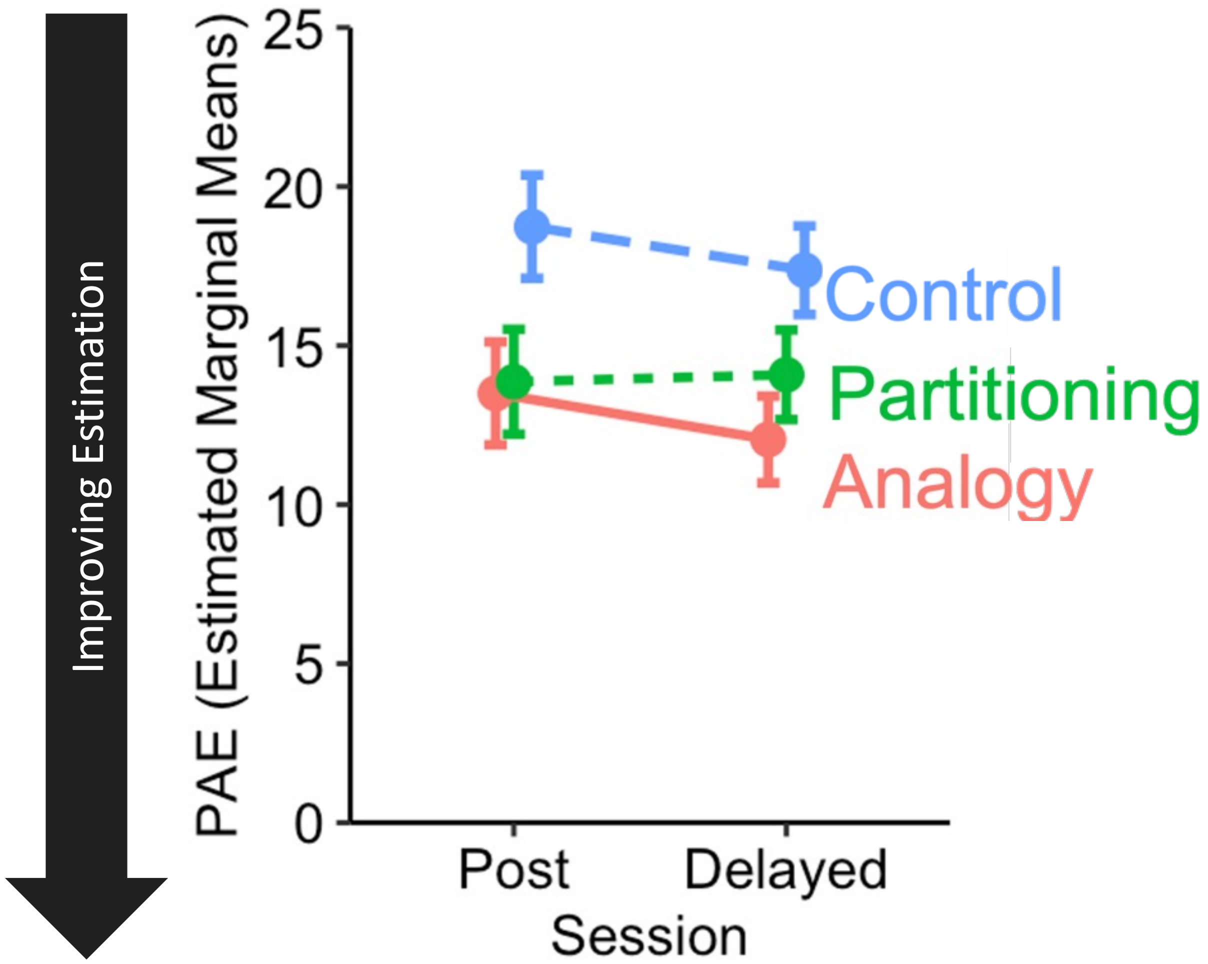
Children with **lower** pretest scores benefitted **more** from receiving the analogy lesson.

A2. Both NL lessons transferred to comparison, but only Analogy group *retained* that learning.

	Outcome	Image	Hypothesis
H1	Number Line Estimation (Learning)	$\frac{6}{7}$ 0 1	Analogy & Partitioning > Control
H2	Comparison (Transfer)	$\frac{1}{7}$ $\frac{2}{6}$	Analogy & Partitioning > Control
H3	Estimation & Comparison with Large-Denominator Fractions	$\frac{15}{42}$ $\frac{8}{11}$	Analogy > Partitioning & Control

Q3. Which is best for large-denominator fractions?

H3. Analogy will be better than Partitioning and Control lessons.



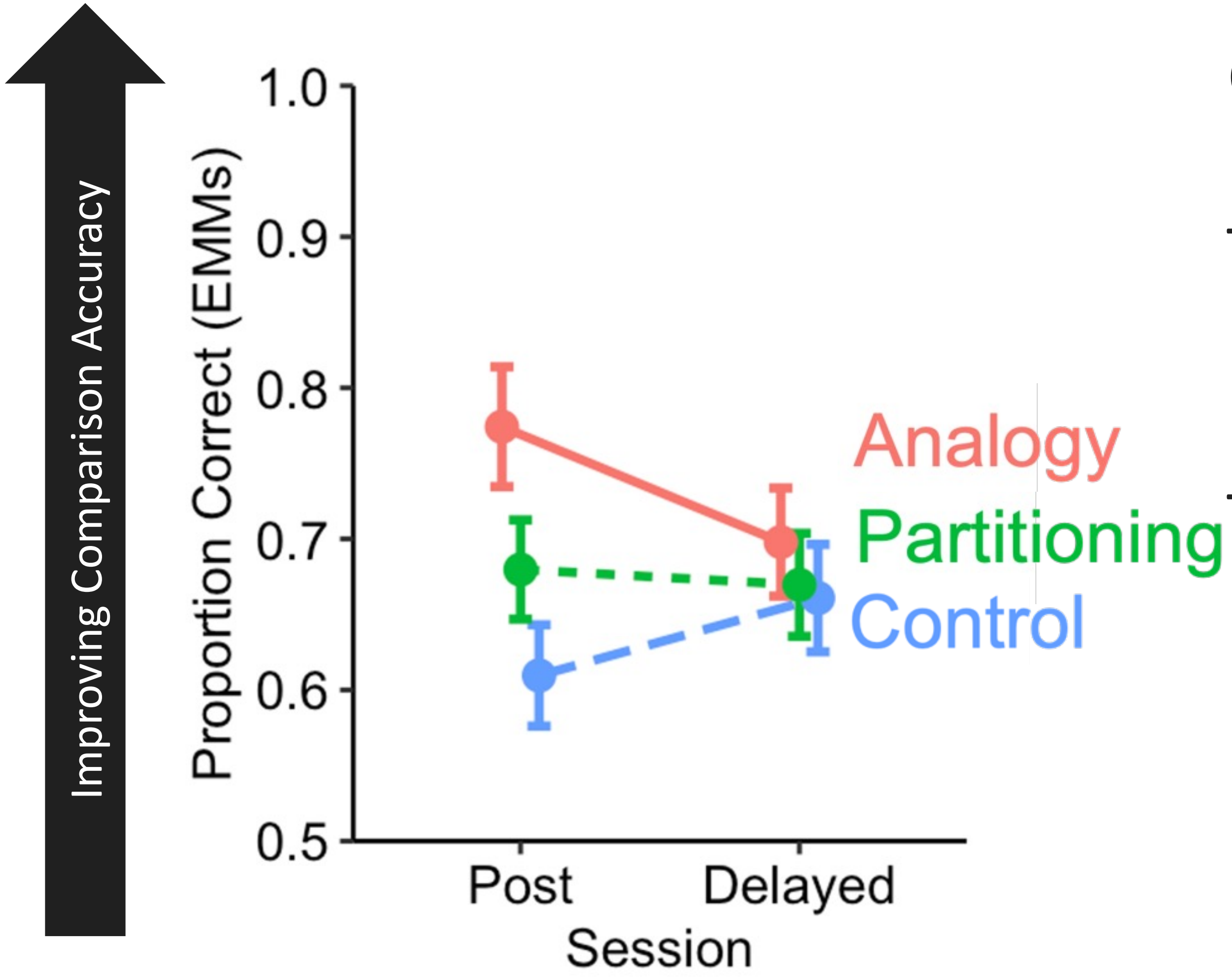
Estimation: Some support for H3

- Both Analogy & Partitioning were better than the control group at *immediate* posttest

- After a one-week *delay*, **only the Analogy group** was better than control group (by about 4%)

Q3. Which is best for large-denominator fractions?

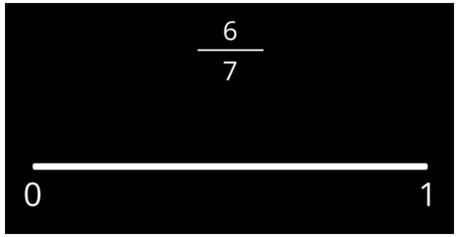
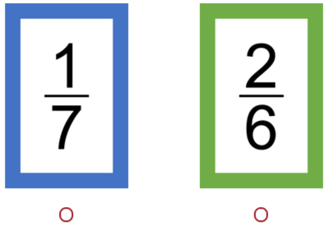
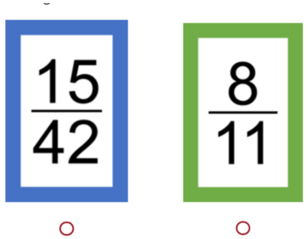
H3. Analogy will be better than Partitioning and Control lessons.



Comparison: Some support for H3

- Only Analogy was better than the control group at *immediate* posttest
- After a one-week *delay*, neither number line group was better than the control.

A3. Some evidence that Analogy was best for large-denominator fractions

	Outcome	Image	Hypothesis
H1	Number Line Estimation (Learning)		Analogy & Partitioning > Control
H2	Comparison (Transfer)		Analogy & Partitioning > Control
H3	Estimation & Comparison with Large-Denominator Fractions		Analogy > Partitioning & Control

Takeaways from Study 1

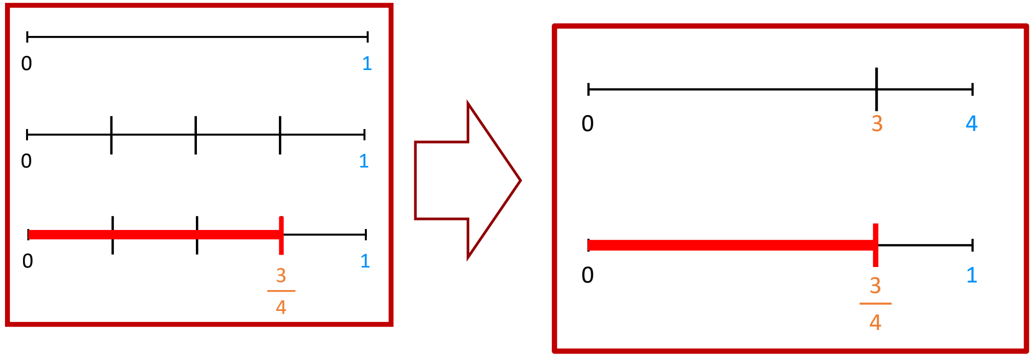
Study 1:
How can we best use **number lines** to support students' fraction **magnitude** knowledge?

1. The **analogy** lesson helped a little more than the partitioning lesson, especially for comparison and for large-denominator fractions.

2. We replicated previous findings that number lines are better for fraction magnitude knowledge, even when lesson is via Zoom.

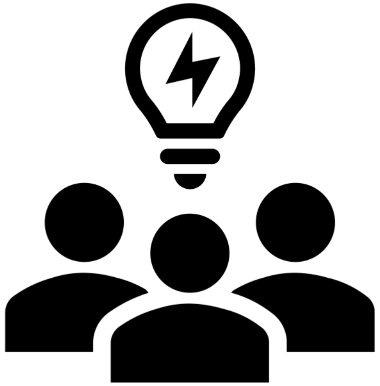
3. Some fadeout from immediate to delayed posttest.

Future Directions for Study 1



Partitioning

Analogy



Improve retention

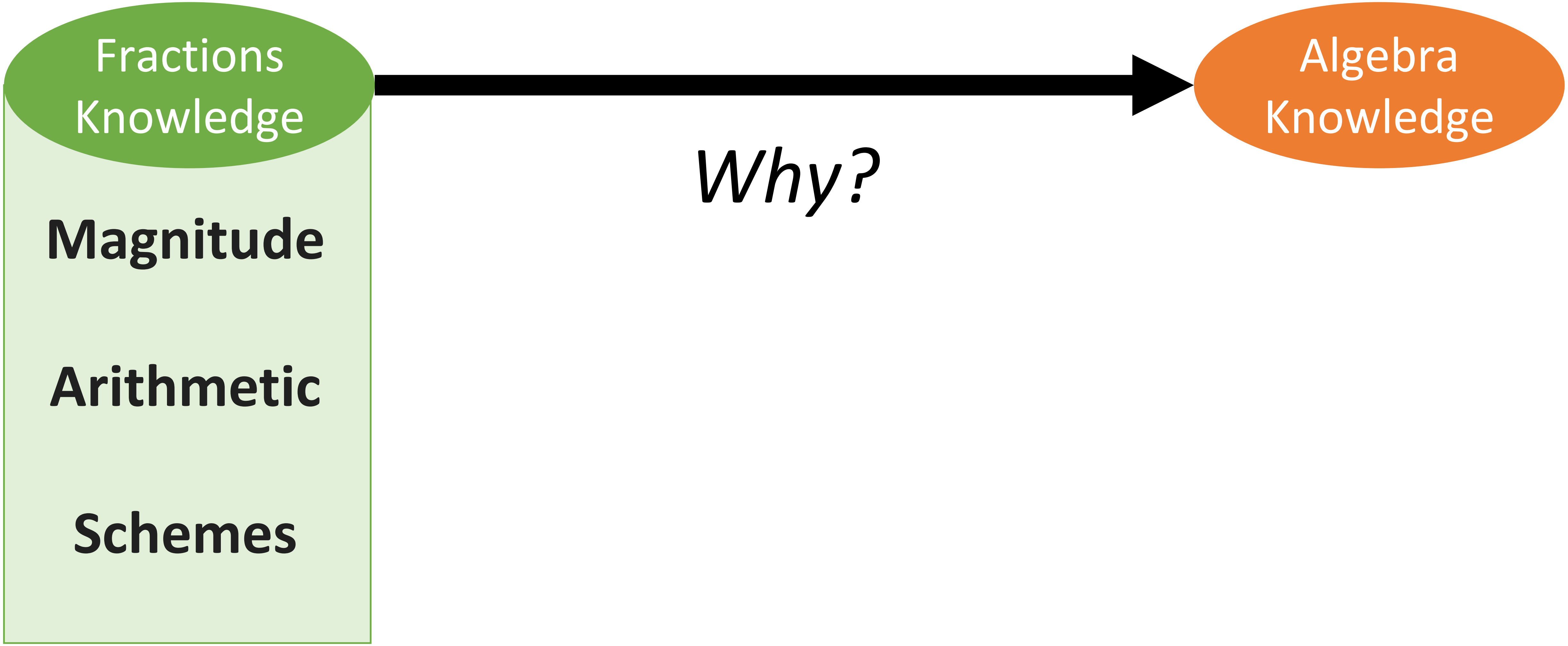
Improve sense-making

Key Questions of this dissertation

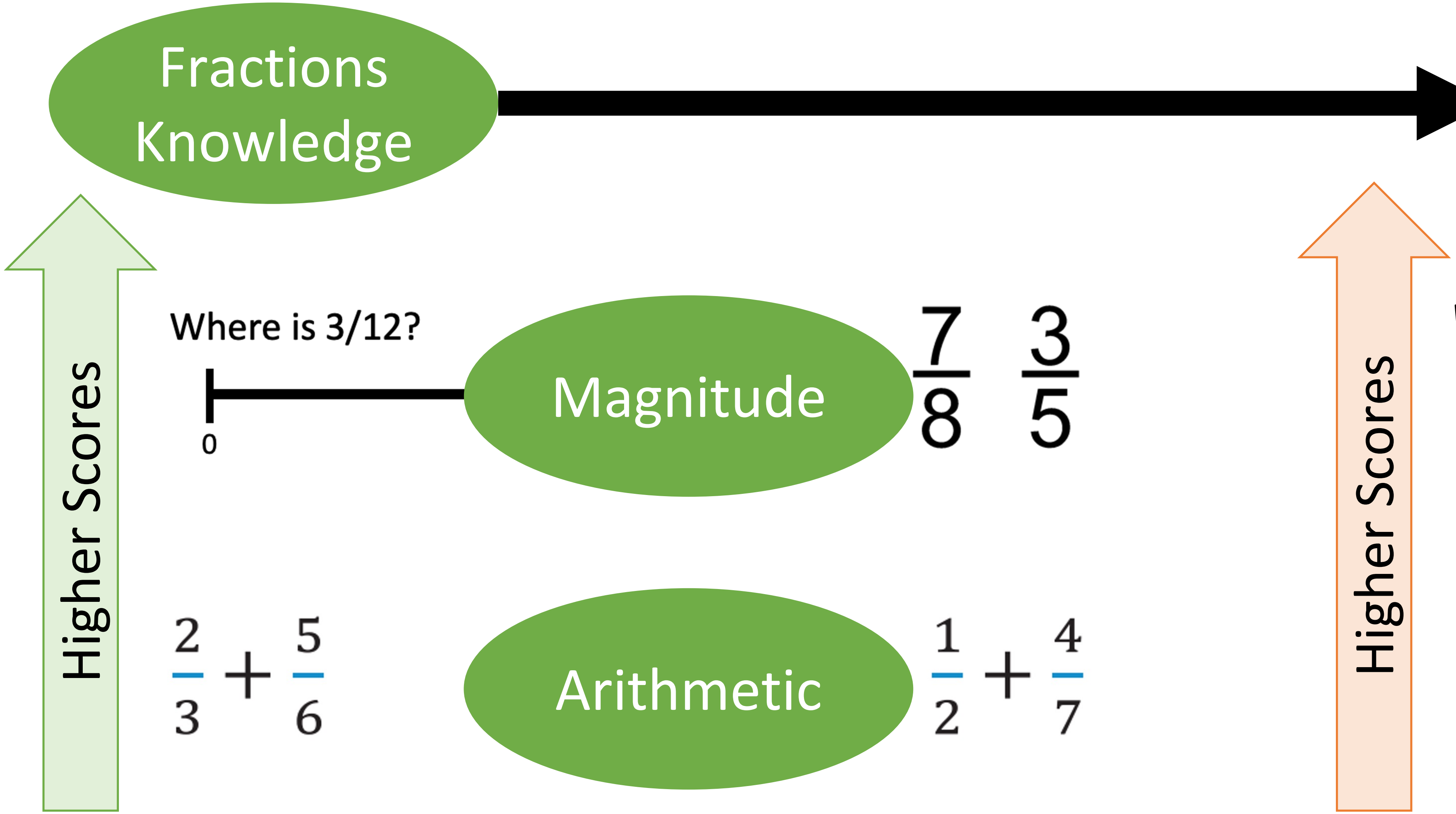
Study 1:
How can we best use **number lines** to support students' fraction **magnitude** knowledge?

Study 2:
What can we learn about **why fractions relate to algebra** from including **multiple measures** of fractions knowledge?

Study 2: Connecting Fractions and Algebra?



Psychology: Fraction magnitude + arithmetic



$3(2x - 1) + 2x = 21$

What is the value of x?

a	*b*
3	7
2	5
0	1

What is the rule?

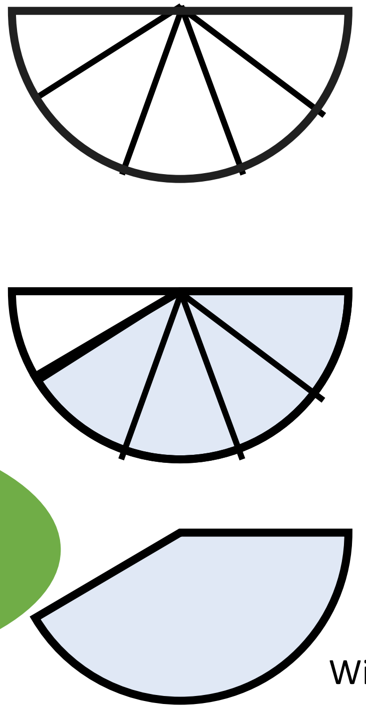
Math Ed: Fraction schemes

Fractions Knowledge → Algebra Knowledge

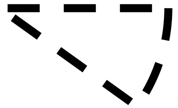
Your piece of pie is 4/5 as big as the piece shown below. Draw your piece of pie.

Partition

Iterate



Disembed



Schemes

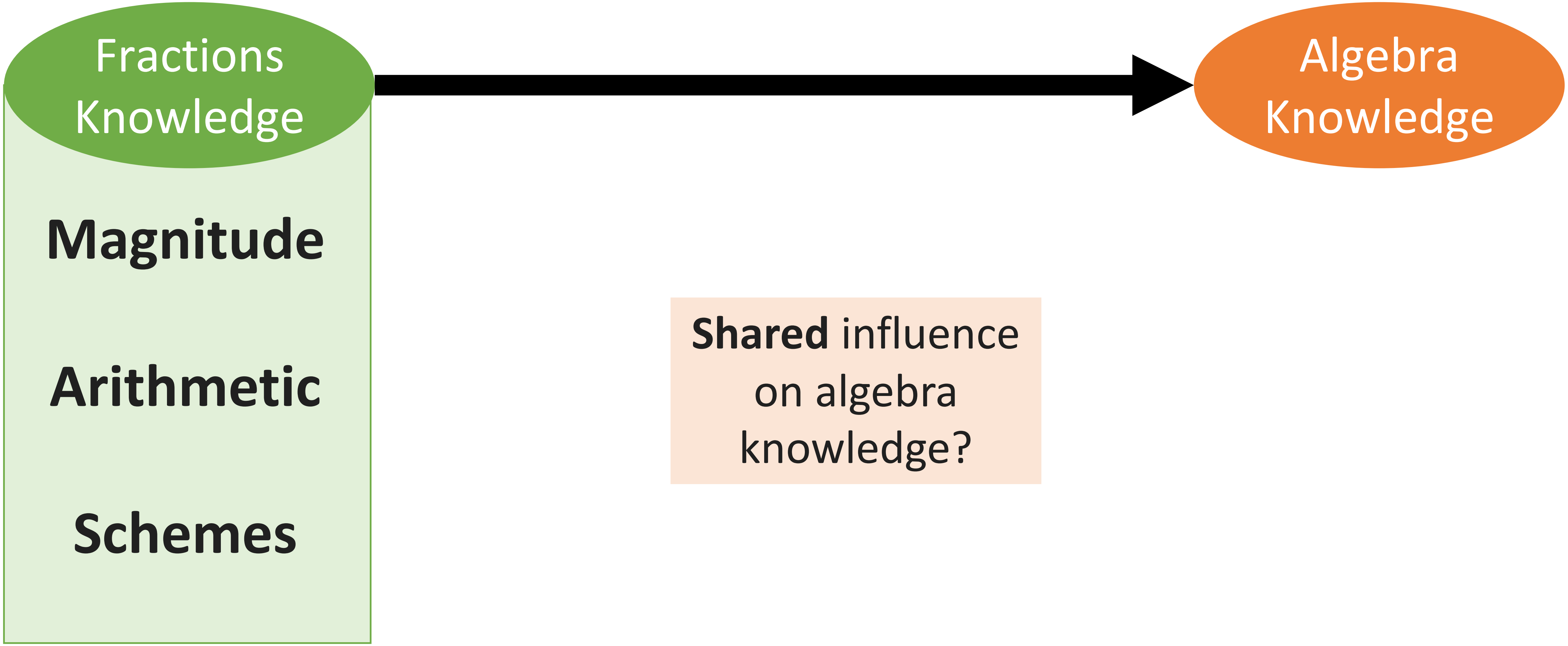
Wilkins et al. (2013)

Stephen's cord is five times as long as Rebecca's cord.
Can you write an equation for this situation?

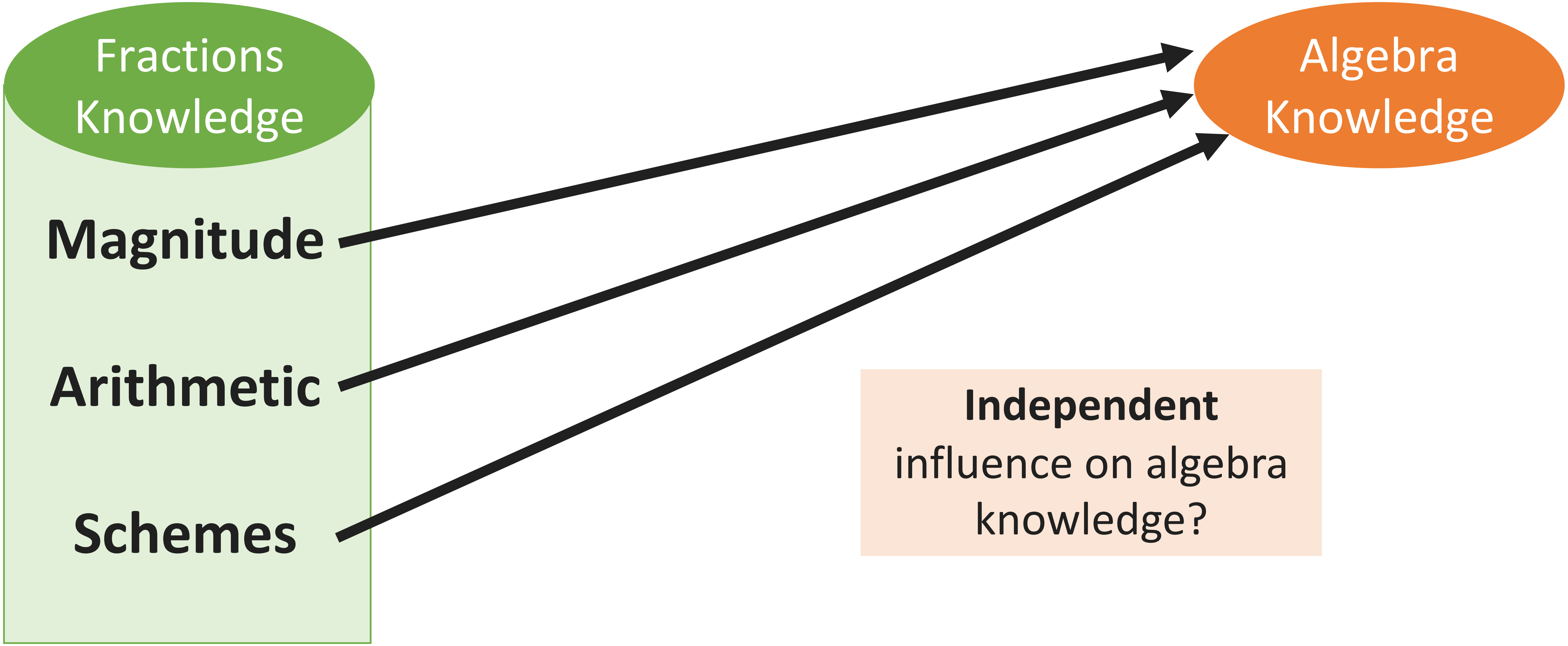
$$S = 5 \times R$$
$$R = S \div 5$$

e.g., Hackenberg & Lee (2015)

Which aspect(s) of fractions are most important?

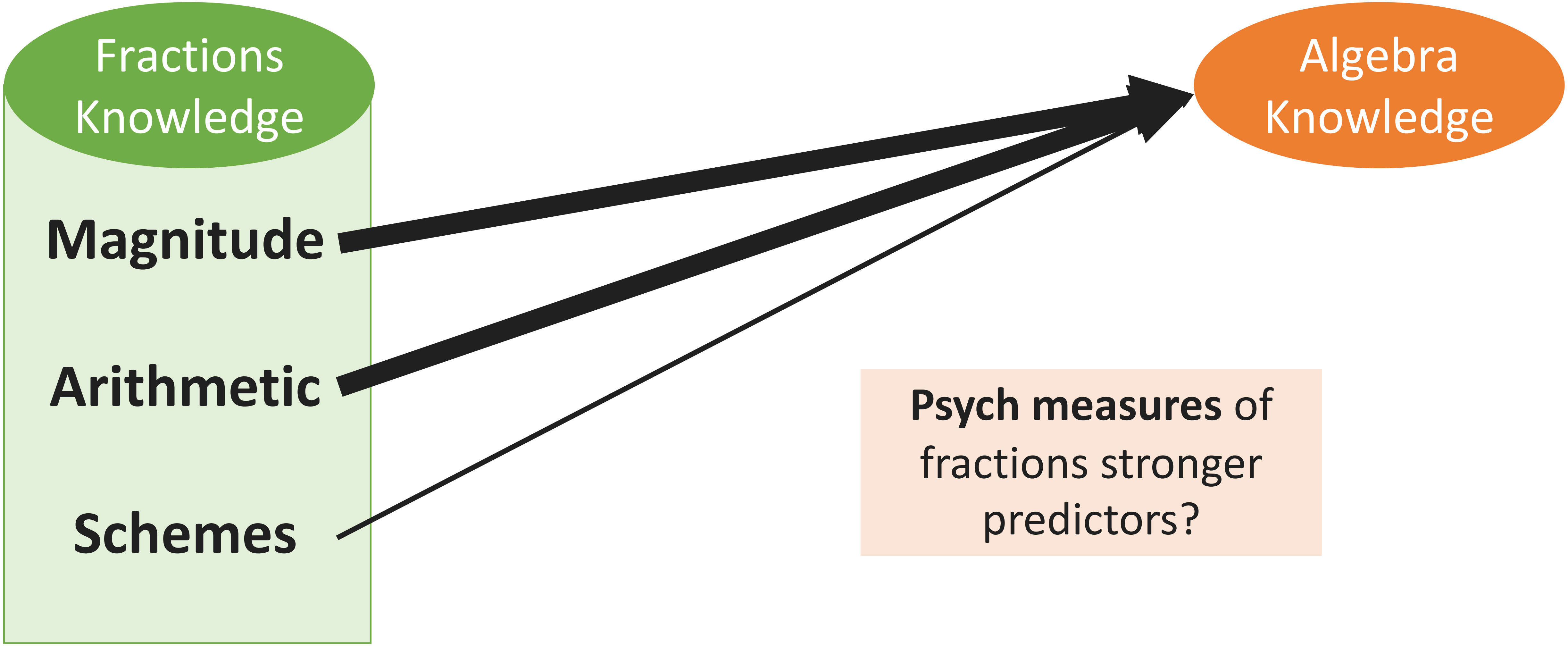


Which aspect(s) of fractions are most important?



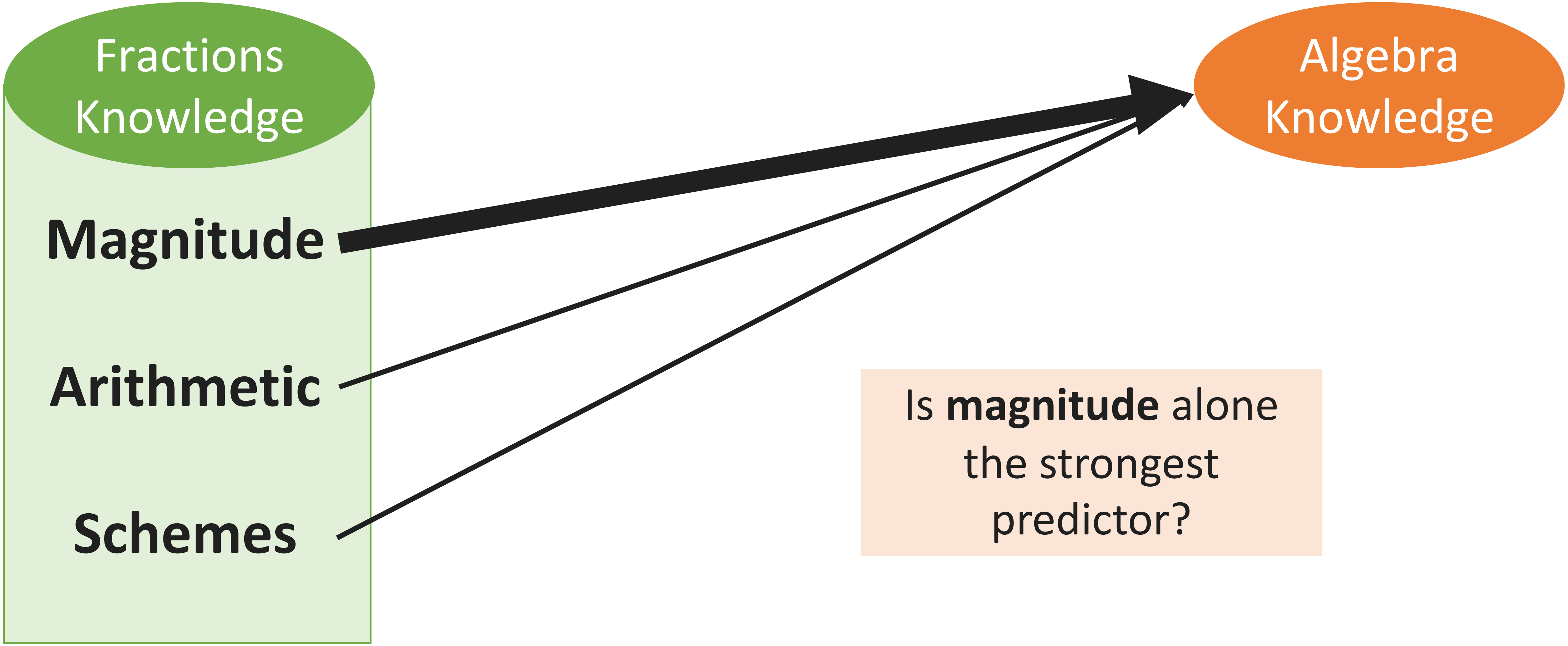
Independent influence on algebra knowledge?

Which aspect(s) of fractions are most important?



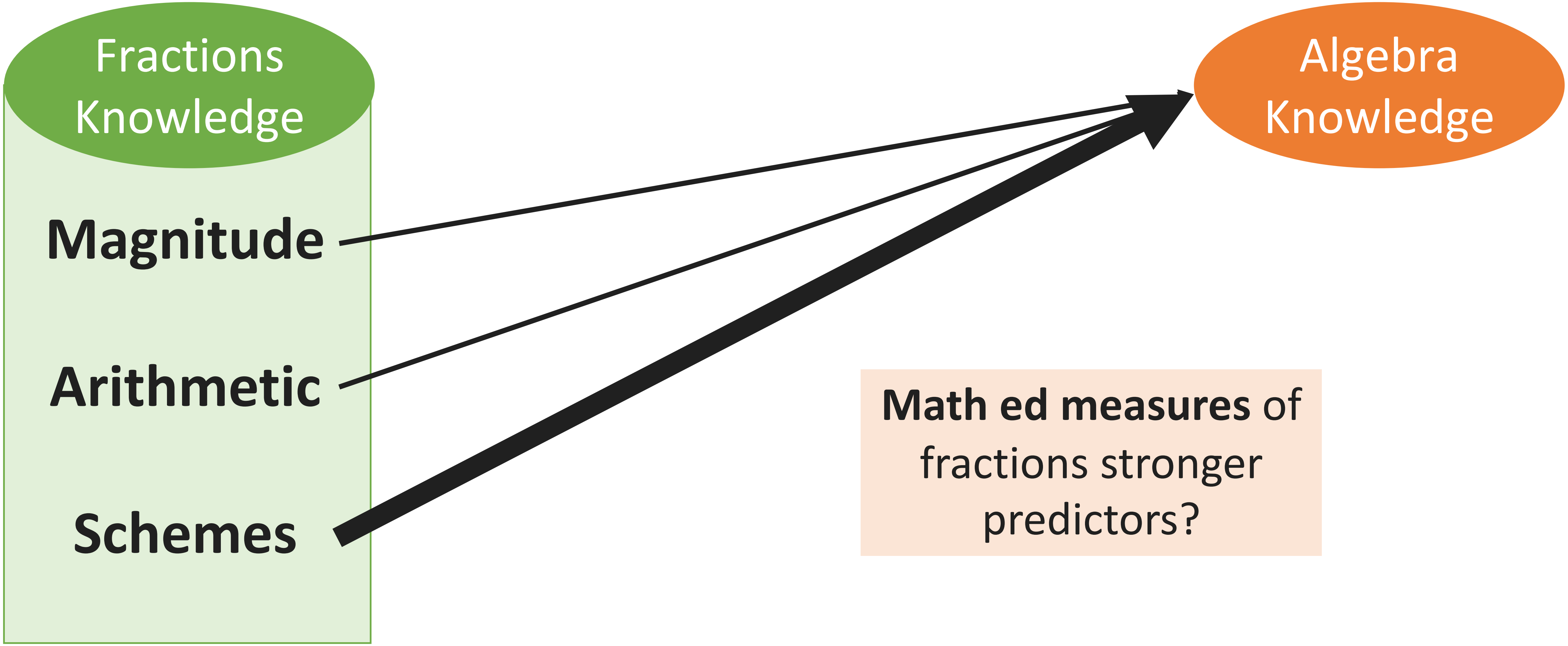
Psych measures of fractions stronger predictors?

Which aspect(s) of fractions are most important?



Is **magnitude** alone the strongest predictor?

Which aspect(s) of fractions are most important?

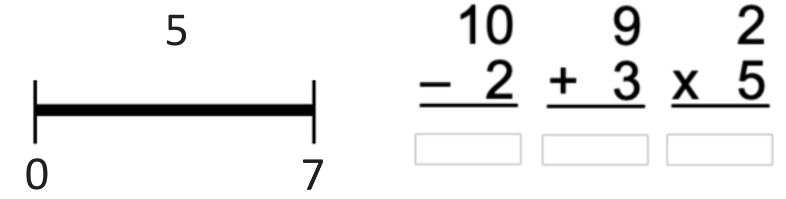


Math ed measures of fractions stronger predictors?

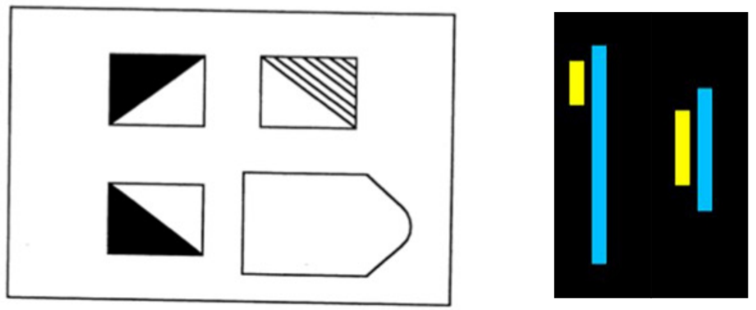
Study 2 Design

8th grade students (N = 59) participated in 3 Zoom sessions.

Covariates



Whole Number skills



General cognitive skills

Math anxiety & WM

Fractions

-Magnitude

-Arithmetic

-Schemes

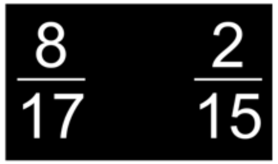
-Units Coordination

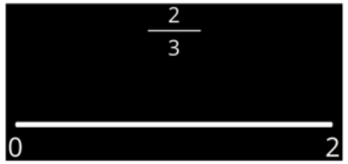
Algebra

45-minute test, Mix of multiple choice & open-ended

Fractions

Magnitude





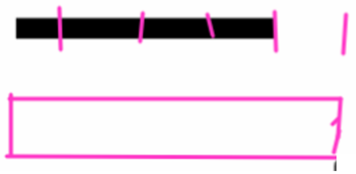
Arithmetic

$$\frac{3}{5} + \left(\frac{3}{10} \times \frac{4}{15}\right) =$$

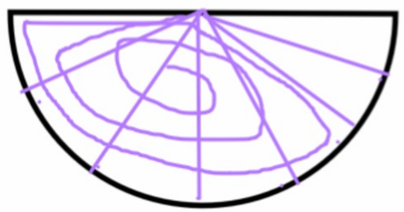
Schemes

3. The bar shown below is 7/3 as long as a whole candy bar.
   Draw the whole candy bar.

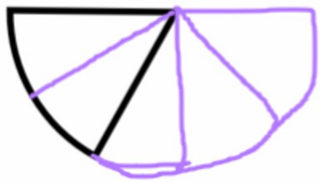
5. The stick shown below is 4/5 as long as a whole candy bar.
   Draw the whole candy bar.



2. The piece of pie below is 7/5 as big as your piece of pie.
   Draw your piece of pie.

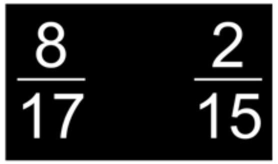


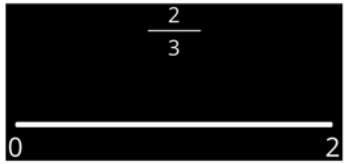
6. The piece of pie below is 2/5 as big as your piece of pie.
   Draw your piece of pie.



Fractions

Magnitude



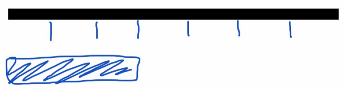


Arithmetic

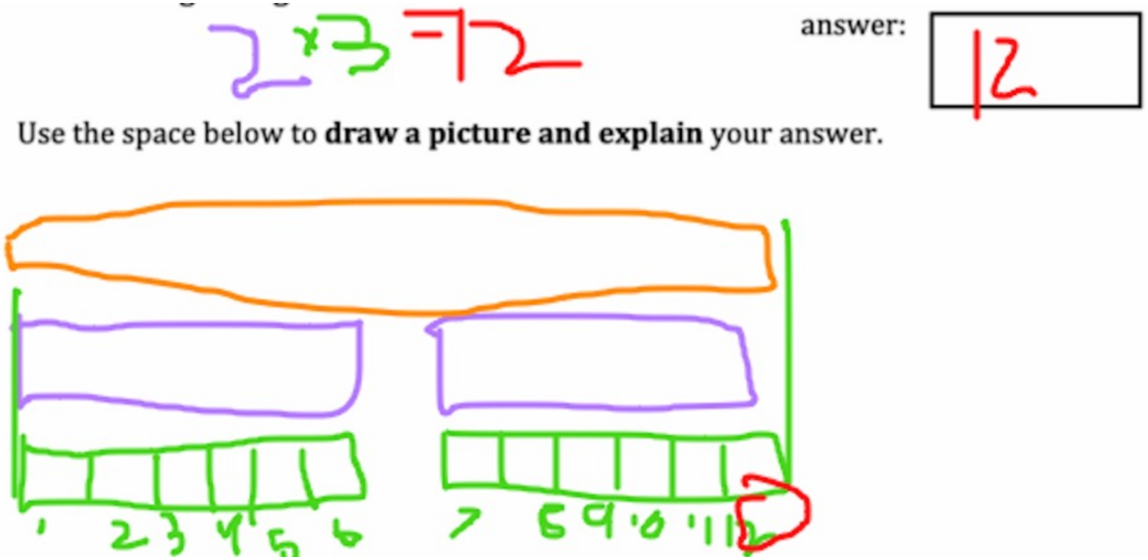
$$\frac{3}{5}+\left(\frac{3}{10}\times\frac{4}{15}\right)=$$

Schemes

3. The bar shown below is 7/3 as long as a whole candy bar.
Draw the whole candy bar.



Units Coordination



Algebra

Which example could represent a linear function?

- ○

x	-3	0	3
y	4	6	8

- ○ $\frac{5}{x} + y = -7$
- ○

x	1	3	5	3
y	4	2	0	-2

- ○ $x + \frac{2}{y} = 4$

Conceptual Knowledge

Below is the beginning of Gabriella's, Jamal's, and Nadia's work in solving x + 7 - 3 = 12 - 2x.

Gabriella's way:	Jamal's way:	Nadia's way:
Subtract 3 from 7: $x + 4 = 12 - 2x$	Add $2x$ to both sides: $3x + 7 - 3 = 12$	Subtract $(7 - 3)$ from both sides: $x = 8 - 2x$

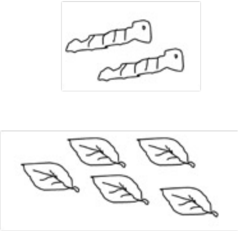
To start solving this problem, which way(s) may be used?

Flexibility

Solve the equation for y. Show your work on paper and type your answer here.

$5(y - 2) = -3(y - 2) + 4$

Procedural Knowledge



A class needs 5 leaves each day to feed its 2 caterpillars. How many leaves would they need each day for 12 caterpillars?

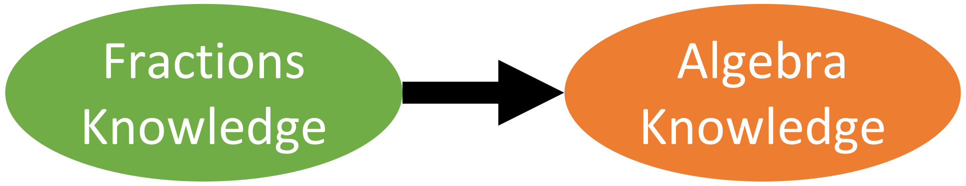
Proportional Reasoning

Overall score

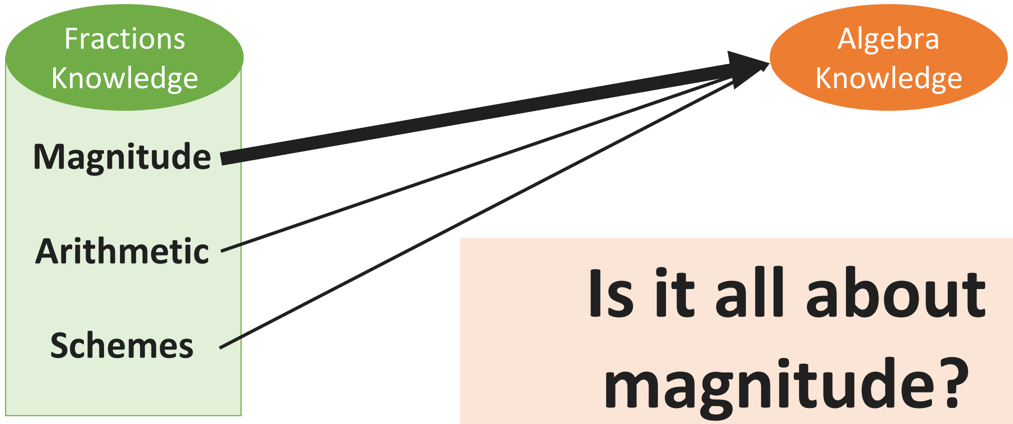
% Accuracy

Study 2 Hypotheses

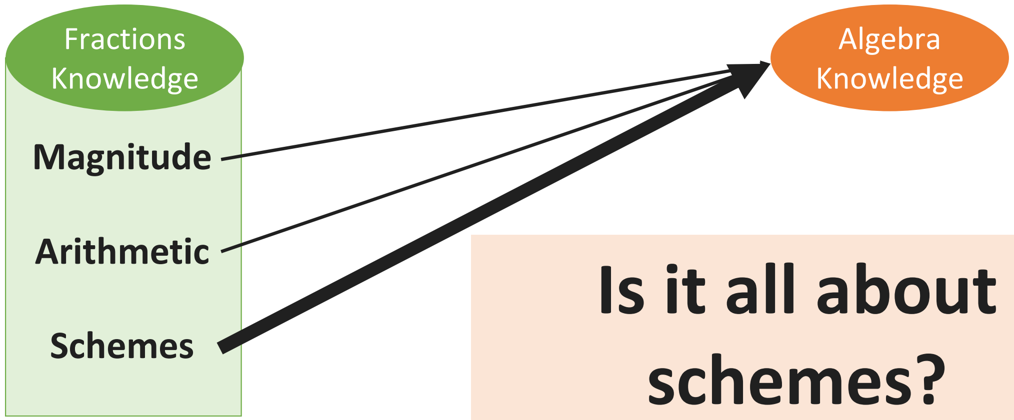
H1. Overall fractions knowledge will predict algebra scores, even accounting for covariates.



H2. Competing hypotheses about which aspect of fractions will be most important…

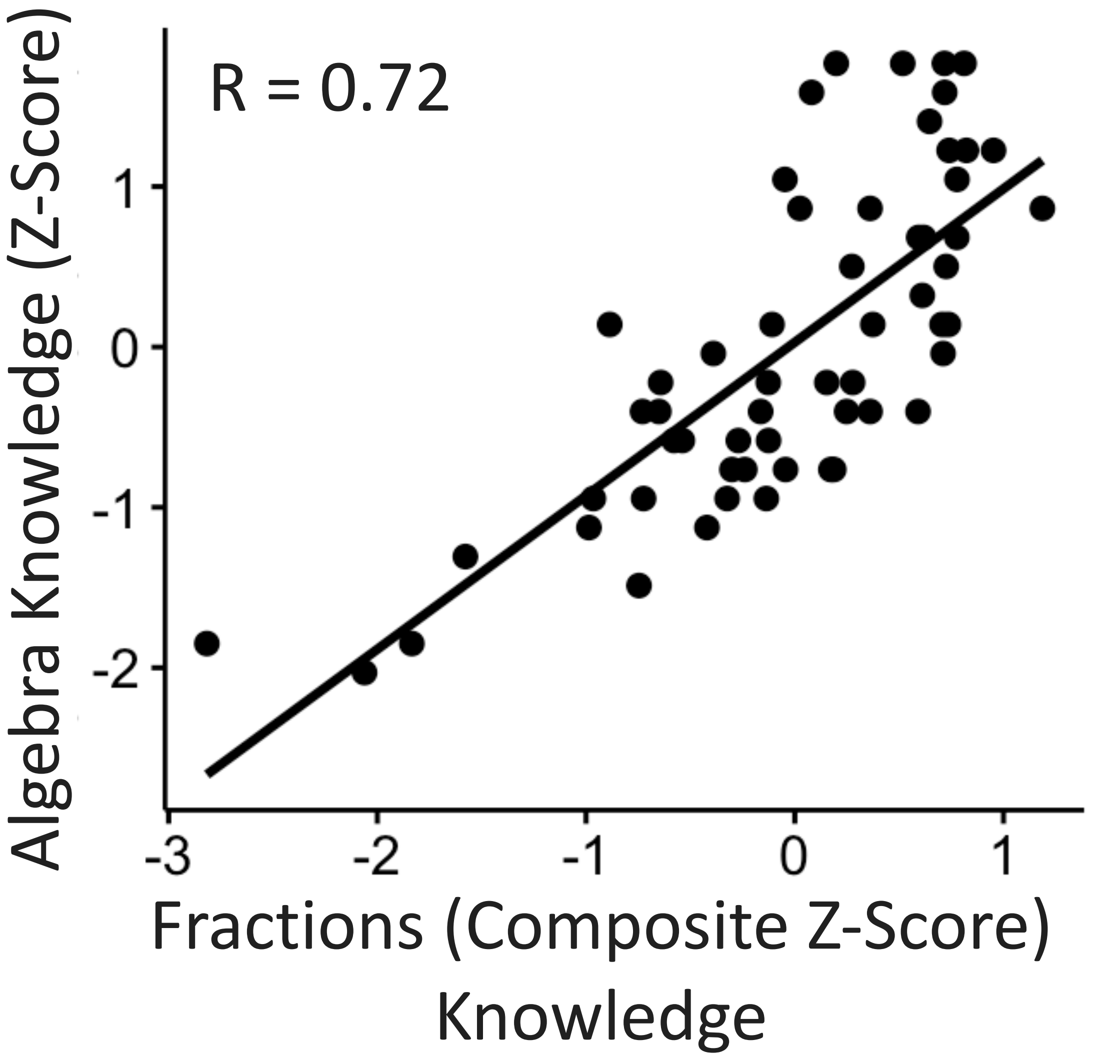


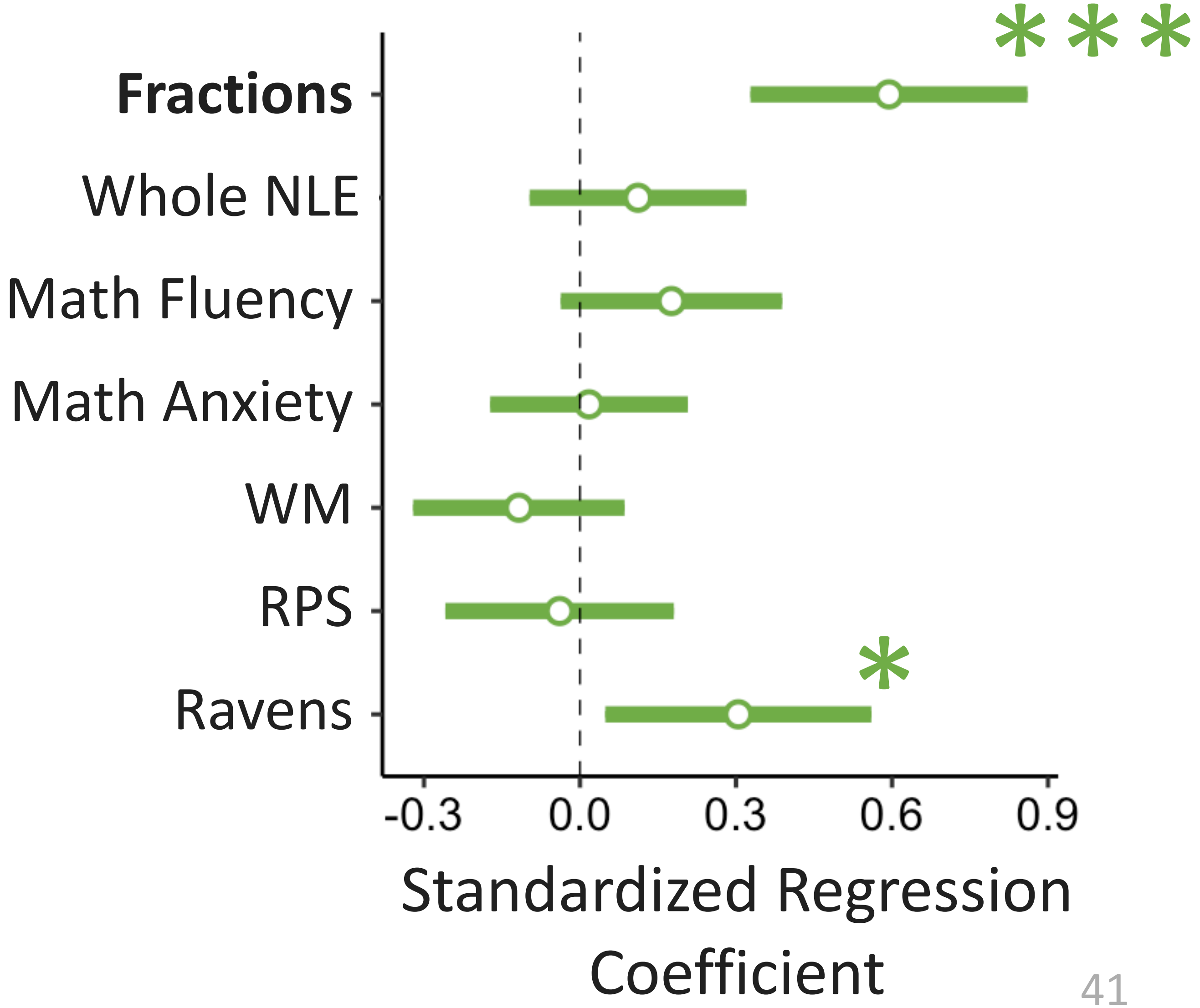
Is it all about magnitude?



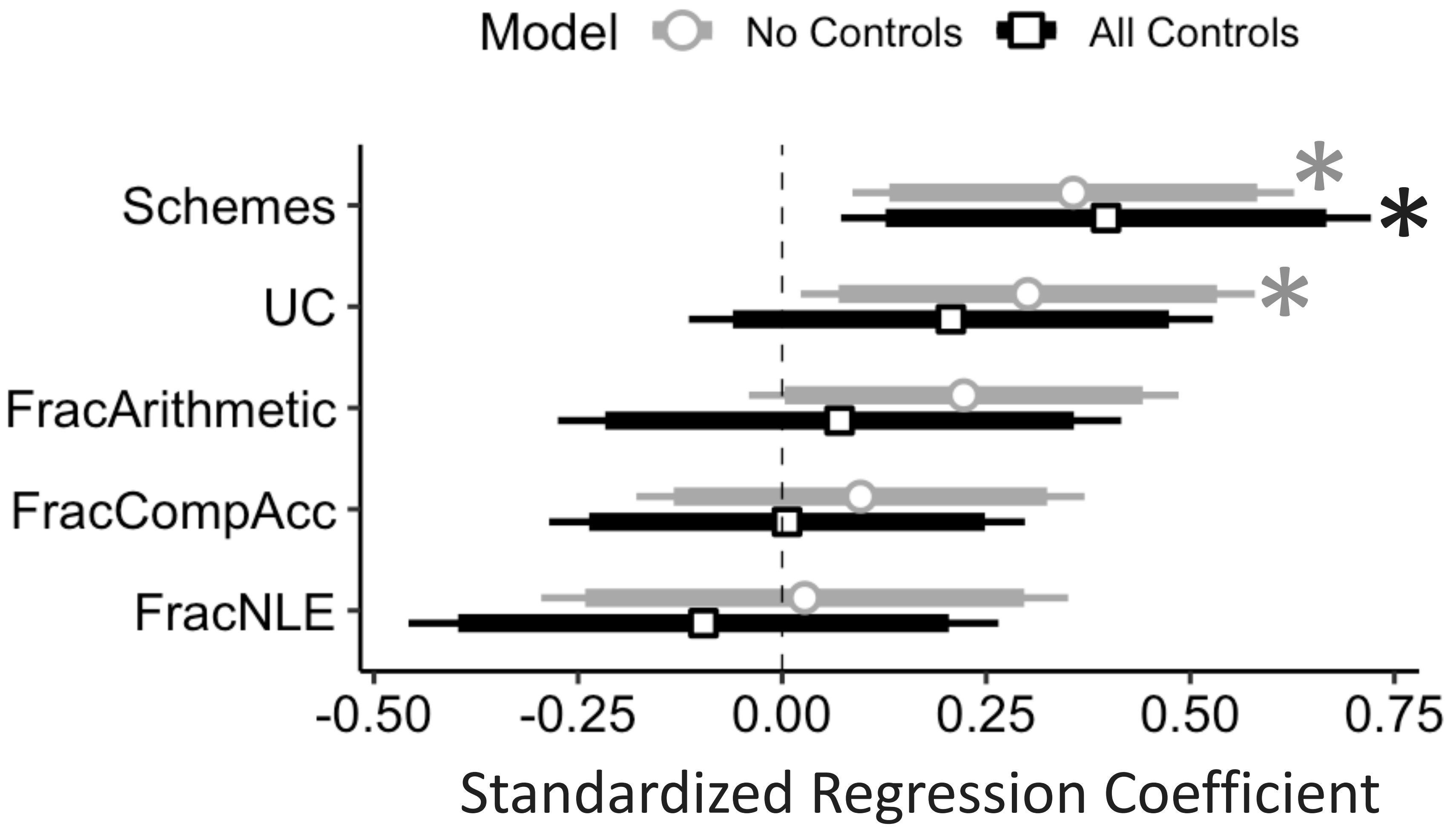
Is it all about schemes?

Overall fractions knowledge predicted algebra.



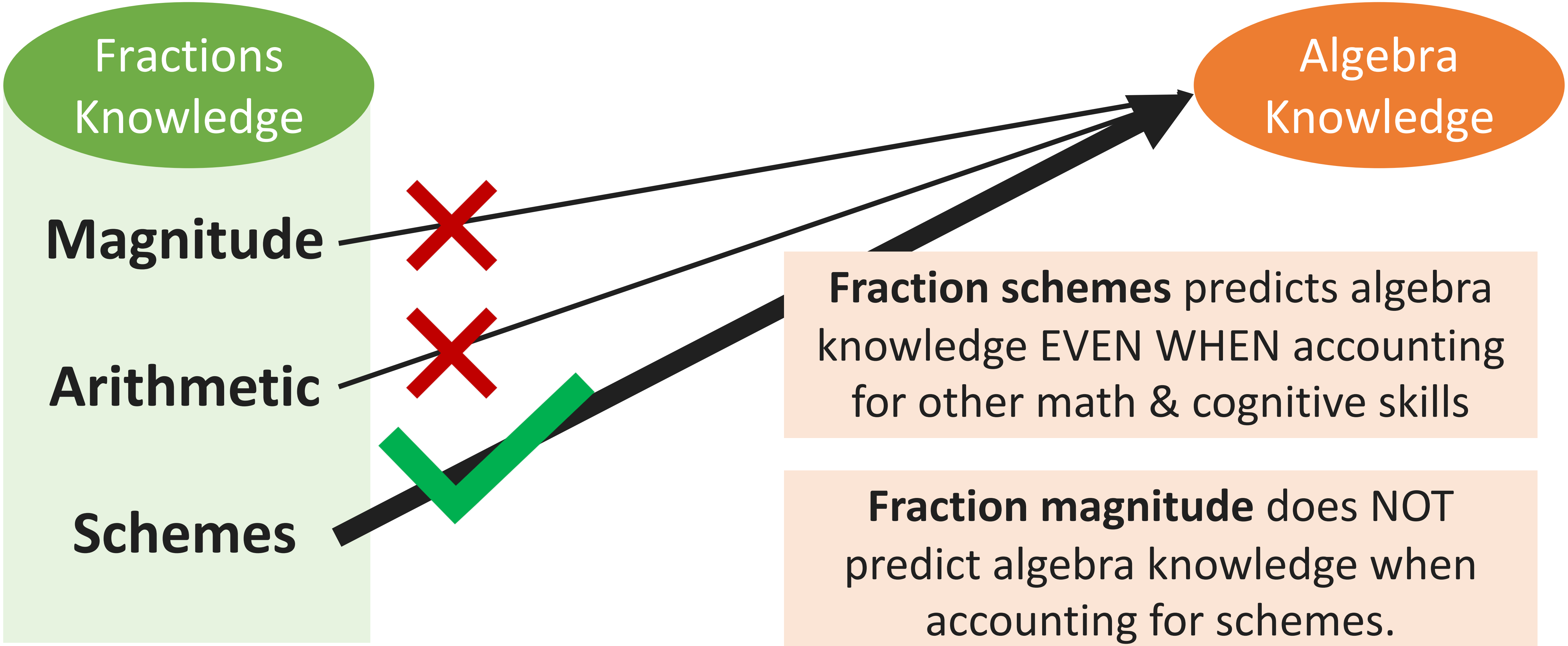


Which aspect(s) of fractions were UNIQUE predictors of algebra knowledge?



After controls, ONLY **fraction schemes** uniquely predicted students' algebra scores!

Takeaways from Study 2



Fraction schemes predicts algebra knowledge EVEN WHEN accounting for other math & cognitive skills

Fraction magnitude does NOT predict algebra knowledge when accounting for schemes.

Future directions for Study 2

Further investigate mechanisms

Efficient recognition of structure/relations?

Improper fractions in particular?

3. The bar shown below is 7/3 as long as a whole candy bar. Draw the whole candy bar.

$$5(y\text{-}2) = \text{-}3(y\text{-}2) + 4$$

Replicate in different contexts

Paper & pencil instead of drawing on screen? Sample with lower math interest?

Summary of Findings

Study 1:

A **15-minute lesson** teaching children to use an **analogy** from whole number estimation to fraction estimation led to improved understanding of **fraction magnitudes**.

Study 2:

Fraction schemes, but not magnitude, **uniquely predicts** students' **algebra** scores when controlling for other math and cognitive skills.

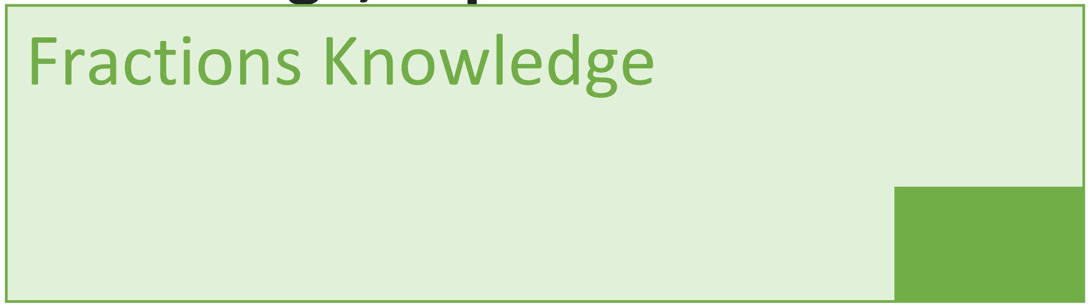
Implications of this dissertation

For future research?

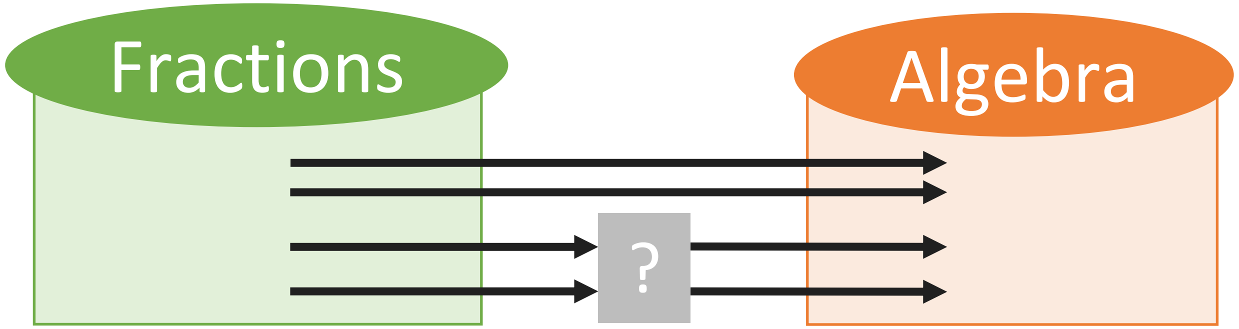
For education and educators?

Implications for future research

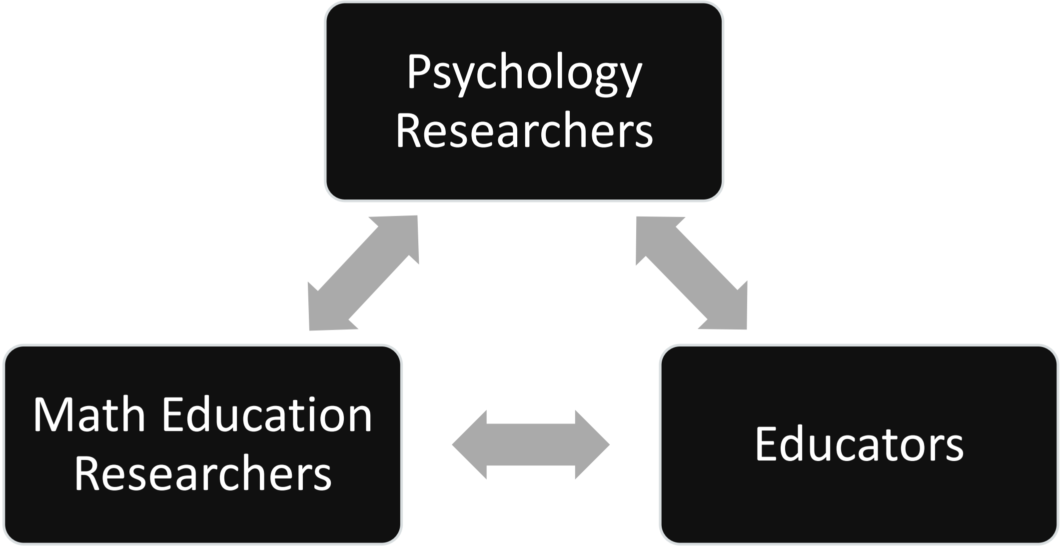
Measure more aspects of fractions knowledge, esp. schemes.



Move toward more *specific* and *actionable* fractions-algebra models.

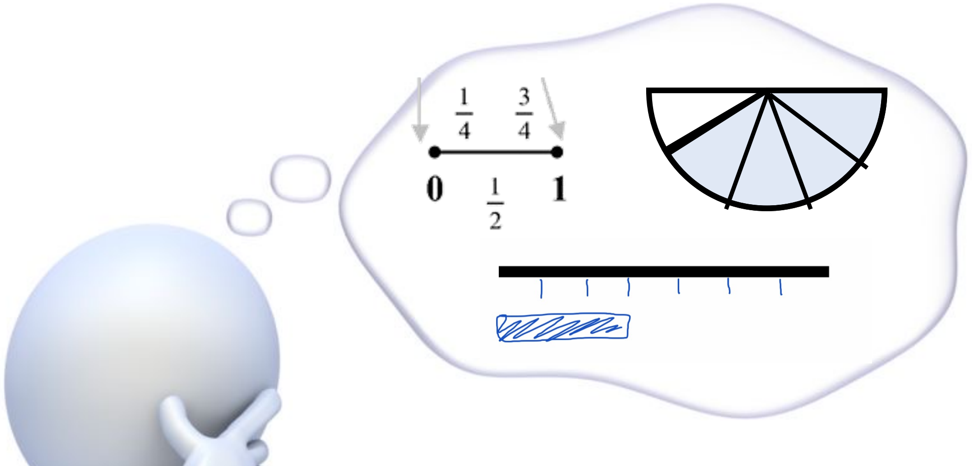


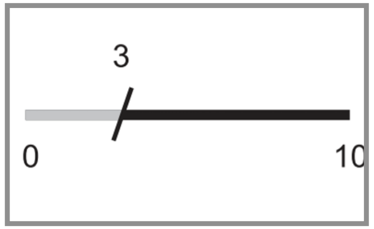
Need for interdisciplinary teams and partnerships with educators.

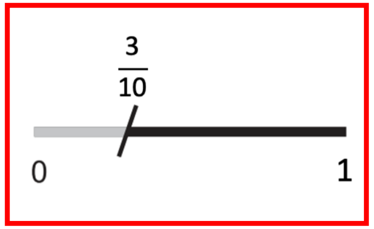


Educational Implications?

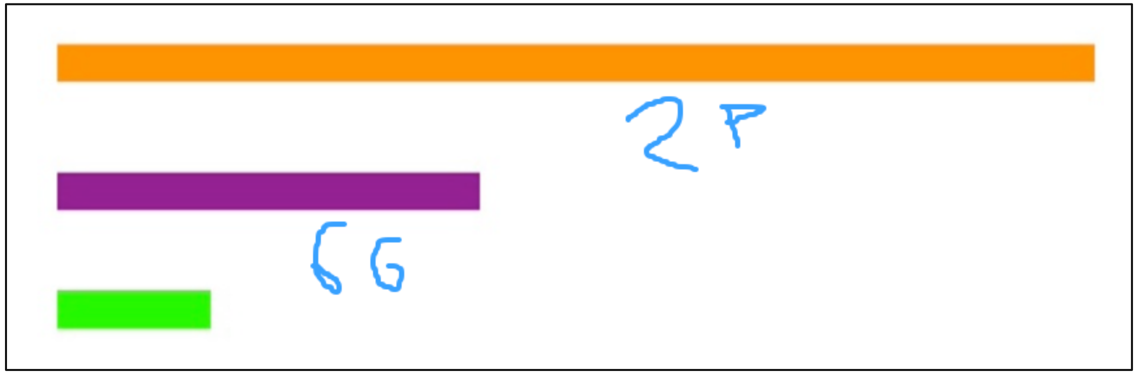
Teach broad fractions knowledge, and leverage intuitions.







Scaffold algebraic thinking during fraction instruction.



a	2	4	8	50	200
b	5	10	20	125	?

Thank you!

Percival Matthews

Ana Stephens

Allison Monday

MELD Mathematics Education Learning & Development Lab

Undergraduate Research Assistants:

Valencia Griffin
Valerie Buroker
Phoebe Miller
Amelia Jensen
Yining Zhang
Rose Eisenmenger

ies

JSMF

WARF Wisconsin Alumni Research Foundation