Title: An Interdisciplinary Approach to Untangling the Fractions-Algebra Relation

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250 word abstract

Fractions knowledge is a unique predictor of algebra performance, beyond other math skills and domain-general abilities (e.g., Booth et al., 2014; Siegler et al., 2012). Although many studies have found an association between fractions and algebra knowledge, it is unclear which aspects of fractions knowledge are related to which aspects of algebra knowledge. Some theories suggest that understanding *fraction magnitudes* supports covariational reasoning, which lays the foundation for thinking about algebraic functions (e.g., DeWolf et al., 2015), whereas others suggest that proficiency with *fraction arithmetic* supports efficient problem-solving in algebra (e.g., Barbieri et al., 2021). Another perspective (e.g., Hackenberg, 2010) suggests that the ability to coordinate hierarchically and multiplicatively nested units undergirds both fractions and algebra knowledge. This study investigated whether fraction magnitude, fraction arithmetic, or units coordination was the strongest predictor of algebra knowledge in 140 students in grades 7–9 in the U.S., when controlling for other related mathematical and cognitive skills. We used separate regressions to predict overall algebra knowledge as well as two subdomains: functional thinking and understanding of equations and equivalence. We found that only fraction magnitude and arithmetic uniquely predicted both aspects of algebra. Our results highlight the value of interdisciplinary work in this area and point to the need for further investigation of the mechanisms driving the fractions-algebra association.

1000 word summary

Algebra is a gatekeeper to higher mathematics and STEM careers (Adelman, 2006; Chen, 2013), and fractions may be a key to the gate. Fractions knowledge is a unique predictor of algebra performance, beyond other mathematical and domain-general abilities (e.g., Booth et al., 2014; Siegler et al., 2012). However, it is unclear which specific aspects of fractions or algebra knowledge drive this connection. Our project investigates competing explanations.

Some theories suggest that understanding *fraction magnitudes* supports covariational reasoning, which lays the foundation for thinking about functions (e.g., DeWolf et al., 2015), whereas others suggest that proficiency with *fraction arithmetic* supports efficient problem-solving (e.g., Barbieri et al., 2021). Another perspective (Hackenberg, 2010) suggests that the connection is driven by *units coordination* — cognitive structures used to reason about hierarchically nested multiplicative relationships which are used in both fractions and algebra. For example, units coordination helps people understand that 24 is 3 units of eight or 24 units of one, that 9/7 is 9 units of one-seventh or 3 units of three-sevenths, and that 4*x* is 4 units of *x* or 2 units of 2*x*.

In previous work, we found that units coordination was more strongly related to algebra knowledge than fraction magnitude or arithmetic, even after controlling for related cognitive and mathematical skills. In the current study, we seek to replicate this finding with a larger sample and to test which of fraction magnitude knowledge, fraction arithmetic performance, and units coordination predict two key domains of algebra: functional thinking, and equations and equivalence.

Method

U.S. grades 7–9 students (N = 140; M_{age} = 13.77; 49.3% male, 47.1% female, 3.6% non-binary) participated in three one-hour sessions on Zoom. The sample was 73% White and about 60% higher-SES. Data collection is ongoing, and we will report results with the full sample (300 students) at the conference.

At session 1, students completed assessments of domain-general cognitive skills and whole number knowledge. Students completed tests of fraction knowledge and units coordination in session 2 and algebra knowledge in session 3.

Algebra Knowledge. Students completed a 38-item assessment of algebra knowledge in one 45-minute session. We focus on the subscales of Functional Thinking (11 items) and Equivalence and Equations (15 items).

Fraction Knowledge. Students completed two measures of fraction magnitude knowledge. First, students estimated the position of fractions on unmarked 0–1 and 0–5 number lines (Siegler et al., 2011). Then, students judged which of two fractions was larger in a 50-item comparison task (Matthews et al., 2016). For both tasks, students answered as quickly and as accurately as possible. To measure fraction arithmetic knowledge, students completed a 12-minute timed test of 19 open-ended fraction arithmetic problems (Kalra et al., 2020).

Units Coordination (Norton et al., 2015; Hackenberg, 2013). Students completed 10 open-ended items to measure units coordination. Seven items showed three bars of differing lengths and required students to coordinate given relationships to determine how many times a smaller bar would fit inside a larger one (see Figure 1). Three items did not include visual representations. For all problems, students explained their reasoning aloud and/or by drawing on the screen. Responses were coded for accuracy and for indicators of Stage 1, 2, or 3 reasoning (following Norton et al., 2015). Coders categorized each student's overall reasoning into one of the three stages. We report results with accuracy here, but results using stages are consistent.

General Cognitive Skills. Students completed the Ravens Advanced Progressive Matrices (fluid intelligence; Raven, 2003), the antithesis subtest of the Test of Relational Reasoning (Alexander et al., 2016), the Backward Digit Span subtest of the Weschler Intelligence Scale for Children (auditory working memory), and the Hearts and Flowers test of executive functions (Wright & Diamond, 2014).

General Math Knowledge and Attitudes. Students completed an 8-item test of whole number long division (Hansen et al., 2015) and the 3-minute timed Math Fluency subtest of the Woodcock-Johnson. They also completed the modified Abbreviated Math Anxiety Scale (Carey et al., 2017).

Results

We ran three separate linear regressions predicting overall algebra scores, functional thinking (FT) scores, and equations and equivalence (EE) scores from the four key predictor variables (fraction number line estimation, fraction comparison, fraction arithmetic, and units coordination). We included parents' education level, grade level, and all available cognitive and general math skills as covariates. Results were similar across all models.

Only fraction arithmetic ($\beta = .40$, $p < .001$) and fraction number line estimation ($\beta = 0.21$, $p = .001$), but not fraction comparison or units coordination, were significant predictors of algebra after controlling for covariates, $F(15, 125) = 22.11$, $p < .001$, $R^2 = .76$. This pattern was the same for the FT and EE subscales (see Table 1). Interestingly, when the fraction scores were removed from the model, units coordination alone predicted algebra beyond all other covariates, $\beta = 0.43$, $p < .001$.

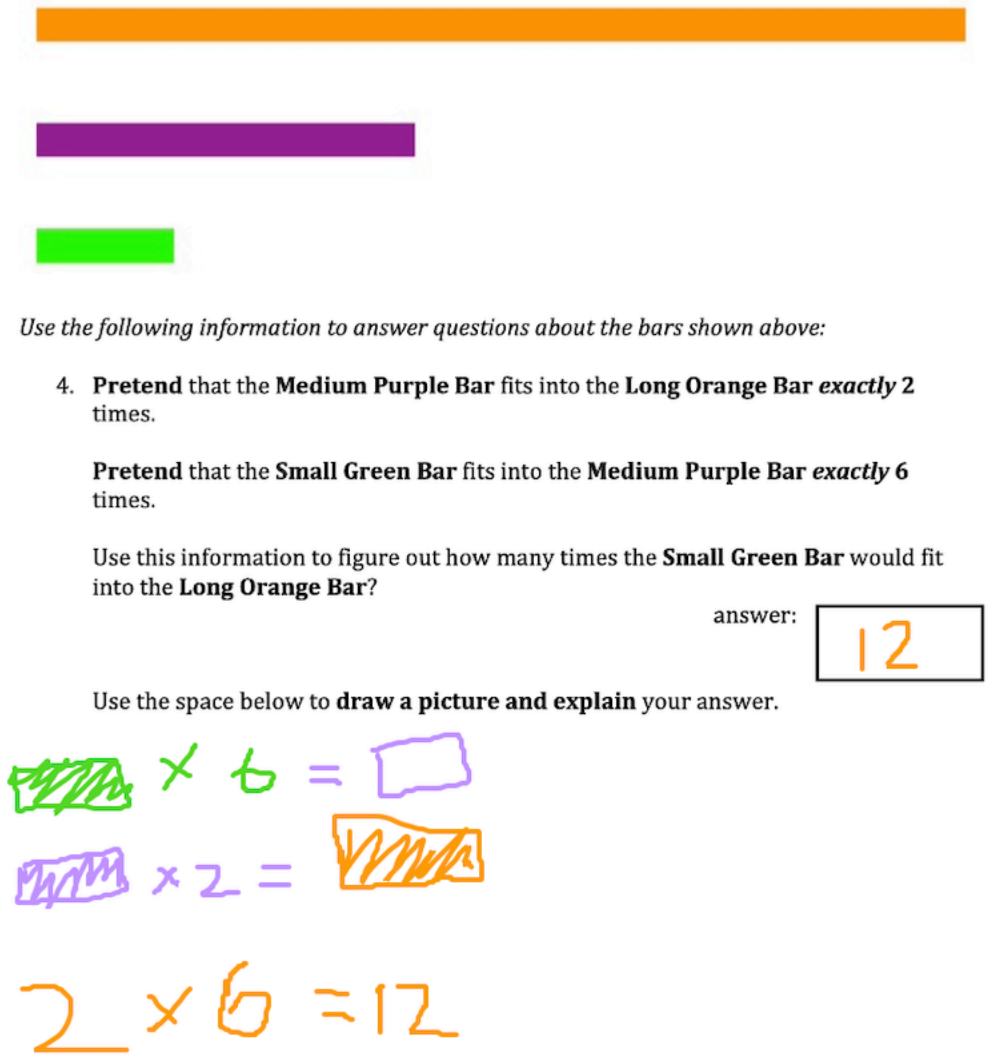
Discussion

Our results strengthen theoretical models of the fractions-algebra relationship. Consistent with past work, we found that fraction arithmetic and magnitude made *unique* contributions to algebra knowledge, even when controlling for units coordination and a robust set of covariates. Strikingly, results were consistent across algebra subtests, which suggests that fraction arithmetic and magnitude may each separately support multiple areas of algebra knowledge. More work is needed to investigate the mechanisms driving these relationships.

This work shows for the first time that units coordination predicts algebra knowledge when controlling for related cognitive and mathematical skills. However, our findings also show that this relationship disappears when we account for fraction knowledge. Our future work will test whether fraction knowledge mediates the relationship between units coordination and algebra knowledge.

More research is needed to directly investigate the educational implications of these findings. Broadly, our findings suggest that instruction that strengthens fraction arithmetic skills and helps students to see fractions as numbers (e.g., Fuchs et al, 2013; Jordan et al., 2024) may prepare students to succeed in algebra. Activities that strengthen units coordination and multiplicative thinking with whole numbers may also set students up for later success with both fractions and algebra.

***Figure 1*. Example Item from Units Coordination with Student Response**



Use the following information to answer questions about the bars shown above:

4. **Pretend** that the **Medium Purple Bar** fits into the **Long Orange Bar** ***exactly* 2** times.

   Pretend that the **Small Green Bar** fits into the **Medium Purple Bar** ***exactly* 6** times.

   Use this information to figure out how many times the **Small Green Bar** would fit into the **Long Orange Bar**?

   answer: 12

   Use the space below to **draw a picture and explain** your answer.

Note. To solve this item, students had to think about three levels of units: the long orange bar contains 2 mediums, each of which contains 6 smalls, so the long orange bar contains 12 smalls.

***Table 1.* Regressions Predicting Algebra Scores**

Predictor	**Overall Algebra** β	***p***	**Functional Thinking** β	***p***	**Equivalence+Equations** β	***p***
Control Variables						
Parent Education	**0.116**	**.022***	**0.164**	**.017***	**0.121**	**.041***
Grade 8 (vs. 7)	**0.133**	**.011***	0.128	.065	0.113	.064
Grade 9 (vs. 7)	**0.153**	**.008****	**0.207**	**.007****	**0.166**	**.014***
Fluid Intelligence (Ravens)	0.090	.105	0.088	.233	0.035	.591
Relational Reasoning (TORR)	0.016	.763	0.033	.647	0.029	.651
Working Memory (Backward Digit Span)	0.019	.707	0.050	.448	0.033	.575
Inhibitory Control (Hearts & Flowers)	-0.052	.355	-0.024	.754	-0.071	.286
Cognitive Flexibility (Hearts & Flowers)	-0.013	.815	-0.077	.316	0.026	.696
Whole Number Division	0.059	.281	0.068	.363	0.052	.420
Whole Number Arithmetic Fluency	0.105	.085	0.052	.525	0.126	.078
Math Anxiety	**-0.119**	**.023***	**-0.139**	**.045***	**-0.124**	**.043***
Predictors of Interest						
Units Coordination Accuracy	0.055	.285	0.061	.375	0.088	.143
Fraction Number Line Estimation (log)[a]	**-0.209**	**.001****	**-0.182**	**.039***	**-0.183**	**.017***
Fraction Comparison Accuracy	0.038	.502	-0.014	.858	-0.010	.879
Fraction Arithmetic Accuracy	**0.399**	**<.001*****	**0.247**	**.006****	**0.365**	**<.001*****
R^2	.76		.55		.66	

Note. All betas are standardized. TORR = Test of Relational Reasoning

[a]Number line estimation percent absolute error was log-transformed to correct for the skewed distribution (Bailey et al., 2012)

*p < .05; **p < .01, ***p < .001